

**AN INVESTIGATION OF THE INFLUENCE OF VISUALISATION, EXPLORING
PATTERNS AND GENERALISATION ON THINKING LEVELS IN THE FORMATION OF
THE CONCEPTS OF SEQUENCES AND SERIES**

by

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Summary

Piaget and Freudenthal advocated thinking levels. In the 1950's the van Hiele developed a five level model of geometric thought. Judith Land adapted the model in 1990, utilising four levels to teach the concept of functions. These four levels have been considered here in the formation of concepts of sequences and series. The origin and relevance of sequences and series have been studied and the importance of visualisation, patterning and generalisation in the instructional process investigated. A series of lessons on these topics was taught to a group of six higher grade matriculation students of mixed ability and gender. Questionnaires related to student progress through the various levels were answered, categorised, graphed and analysed. Despite the small number of students, results seem to indicate that emphasising visualisation, exploring patterns and generalisation and teaching the topics as a reinvention had made a positive contribution towards progress through the various thought levels.

Student Number: 3982009

I declare that **AN INVESTIGATION OF THE INFLUENCE OF VISUALISATION, EXPLORING PATTERNS AND GENERALISATION ON THINKING LEVELS IN THE FORMATION OF THE CONCEPTS OF SEQUENCES AND SERIES** is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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CHAPTER 1

Introduction and overview

1.1 Background to the study

Mathematics is a subject which is frequently held in awe and misunderstood by students. Many matriculants study rules merely for examination purposes without really appreciating their worth or relevance. In the topic of sequences and series, students are often presented with sets of rules without being aware of their origin or purpose. This causes them to memorise formulae without being conscious of their true meaning and often leads to errors due to lack of understanding. It would seem that experiencing the development and being made aware of the relevance of the topic of sequences and series could help to promote appreciation of these topics.

The theories of Piaget, van Hiele and Freudenthal support the concept of levels of learning in mathematics. The levels of thought involved in the development of mathematical concepts by matriculation students could give some insight regarding the development of students' mathematical thinking. The effect of emphasising visualisation, exploring patterns and generalisation appear to be relevant in this regard. Consequently a series of lessons on sequences and series in which learning levels are taken into account and visualisation, patterning and generalisation are promoted follows in Chapter 4.

Visualisation is the first of the three factors which will be emphasised in order to promote advancement from one level to another. Very little attention is paid to this aspect in the teaching of sequences and series. However, it could form a very vital part of the introductions to these topics, facilitate movement from one level to another and enhance thought processes while working at the various levels. Numerous visual examples will be provided for students and they will be encouraged to form mental pictures of the topics they study.

Since there is an abundance of patterns in sequences and series, exploring patterns will play a significant role in the lessons. Patterns will be the main theme of the first lesson in order to stress patterning, encourage students to be aware of patterns and to look out for them throughout the study. Patterns too will be utilised to promote advancement of students through thought levels and encourage efficient performance at each level.

Generalisation is the third important point which will be emphasised in the lessons in Chapter 4. As a result of the visual examples and patterns provided for students, they will be led to generalise their results in order to establish formulae and general results. This they will be encouraged to do by themselves after studying the material provided and having group discussions. Students will be made aware of the need to prove their findings. They will be provided with results that appear to be generally true but made aware that, even though some may appear to always hold, they actually do not. The method of proof by mathematical induction will be introduced here as a high level activity. This will provide students with a way of verifying some of the results they have established on their own in a universally acceptable manner and appreciating that these are in fact always true.

The main theme in the lessons of Chapter 4 will be to teach the topics of sequences and series in a manner which Freudenthal would describe as a reinvention. The students will be taken through the various levels of learning by the provision of material emphasising visualisation, patterning and generalisation. Their responses to the lessons will be analysed in Chapter 5 while in Chapter 6 conclusions will be drawn and recommendations made.

1.1.1 Orientation

The levels of thought involved in the development of the concepts of sequences and series seem to be significant. Research done over the years has shown that there are different levels of thought and that people need to pass through these levels in the development of their mathematical thinking. The famous Genevan, Jean Piaget, was born in 1886 in

Switzerland and died in 1980. He was interested in the mistakes children were making and developed insight into the way in which children think. Zevenbergen (1993) notes how he became one of the most influential people in education especially as far as mathematics and science are concerned. He believed that children pass through different stages of thinking in their development from infancy to adolescence. The stages are referred to as sensorimotor, preoperational, concrete operational and formal operational. Although these stages generally apply to certain ages, there might be a range of ages at which they occur. Nevertheless, he believed that they are ordered in time and that knowledge is acquired naturally by means of experiences in which the individual becomes actively involved. These stages of development in mathematics education are significant and Zevenbergen (1993) remarks that they are so entrenched into the construction of mathematics curricula that it is hard to imagine it as being otherwise.

In the nineteen-fifties Pierre van Hiele and his wife Dina van Hiele-Geldof of the Netherlands began to concentrate on the development of geometric insight in their pupils. In 1951 they developed their model in which they expressed the belief that students progress through different levels of thought in geometry, called levels 0 to 4 or levels 1 to 5. At the beginning children display a Gestalt-like visual approach (Gestalt psychology proposes a correspondence between what we see and hear and brain processes) but then pass through increasing levels of sophistication including analysis, informal deduction, formal deduction and rigour. Whereas Piaget's belief is that age or biological maturation is important, the van Hieles believe that advancement from one level to another depends on instruction and that instruction must match a student's level of development in order for learning to occur.

Land (1990) considered the application of van Hiele's theory to the teaching of functions. She consulted with van Hiele and it was agreed that four levels should be considered in the teaching of this algebraic topic as the fifth level would be too high to attain in this case. The levels were considered to be the basic visual or pre-descriptive level, the descriptive level, the theoretical or theoretical informal level and the deductive or theoretical formal level. The

topic of sequences and series will be taught in Chapter 4 by taking the thinking levels described by Land (1990) into consideration.

Van Hiele was a student of Freudenthal, who claimed that if practice in the classroom

. . . is more efficiently structured, there is every prospect that more pupils will sooner attain higher levels of dealing with it. Observing that the learning processes have revealed steps and intermediate steps, levels and degrees, so a new design can show stricter programming in the sequence of the structured examples in a way that all necessary and accidental learning processes are strongly suggested to the learner (Freudenthal 1978: 182).

Thus Freudenthal also claims that the learning process is structured by levels and that the activity of the lower level becomes the object of analysis at the higher level. He observes:

"Mathematics is an organising activity. By thinking about organising one enters a higher level in the learning process. This thinking is called reflection" (Streefland 1993: 38). Freudenthal believes that mathematics involves growing, developing, looking for problems and organising subject matter. However, since once these results are placed in a list, the original character seems to be lost, he expresses concern that the sources of insight be kept open in the teaching - learning process. He complains that the

. . . 'smoothed over' end product of the historical learning process was the point of departure in education and that beginning with definitions, axioms and other formal fitness is in short the genesis in reverse as a didactical principle (Streefland 1993: 122).

Referring to Hans Freudenthal, Goffree remarks how "from the very start, HF lifted the level theory across the boundaries of geometry" (Streefland 1993: 37). For example, referring to the teaching of mathematical induction as

. . . a subject hard to teach and inaccessible for most pupils, Freudenthal indicated that this was because people always began on far too high a level and did not give the pupil the opportunity to go through the previous, lower levels of the learning process (Streefland 1993: 37).

This point will be relevant in the teaching of mathematical induction in Chapter 4.

As advocated by Freudenthal, the approach in the method of teaching sequences and series here is not to teach the students a piece of formalised mathematics but rather to accompany them in a learning process in such a way as to make them conscious of it. In each lesson the content would not merely be the topic but the understanding of the meaning of formalising

and algorithmising in mathematics. Algorithms are of major importance in mathematics but the danger lies in teachers being tempted to teach them without allowing pupils to reinvent them. However, if the natural development of mathematical language in phases of abstraction or formalisation is shown, pupils are better able to understand the process. In this way they go through various learning processes to ascend to the levels of the language required rather than being immediately served with the finalised linguistic form. Students would be encouraged to appreciate the end results by being involved in the process of arriving there.

Piaget, van Hiele and Freudenthal all recognise the importance of visualisation in mathematics and the inclusion of visual representations in the introduction of new concepts. The first of the three levels of Van Hiele's later model may be described as the visual level. It is based on non-verbal thoughts and observations and language suitable for describing relevant properties is developed for when the descriptive level is reached. In subsequent lessons pupils will be provided with appropriate materials, encouraged to reflect on their findings and be taught the relevant symbolism and terminology. Freudenthal describes as an instructional principle to make use of "... concrete, if possible, visual situations" (Freudenthal 1991: 56).

Various processes involved in studying the topic of sequences and series will be considered here. Moodley (1992) describes the formalistic approach to mathematics education as being one which involves acquiring a body of knowledge by using certain algorithmic skills as well as deductive reasoning ability. However, Anna Sfard indicates that the processes approach reflects the idea that "... in the research in mathematical learning and teaching, mathematics must be seen not only as a ready-made product, but also as a process" (Sierpiska & Kilpatrick 1998: 507). The processes approach views mathematics as consisting of various processes. Two of the processes involved in dealing with sequences are described by Hargreaves, Threfall, Frobisher and Shorrocks-Taylor as being "searching for patterns in a sequence" and "generalizing a rule in words and/or algebraic symbols"

(Orton 1999: 81;82). These processes will form part of the investigation in Chapter 4. Proof by mathematical induction in order to verify various results of findings will also be included.

1.2 Problem statement.

In the teaching of sequences, series and mathematical induction, does emphasising visualisation, exploring patterns and generalisation promote progress through the van Hiele thought levels as related to algebra?

1.3 Aims and objectives

1.3.1 Aims

- To highlight the relevance and background of the topics of sequences and series in mathematics and mathematics education.
- To investigate the importance of the learning levels described by Piaget, van Hiele and Freudenthal in the teaching of sequences and series.
- To determine the roles played by visualisation, patterning and generalisation in the advancement of pupils' thinking to subsequent levels in the teaching of sequences and series.

1.3.2 Objectives

Chapter 1 Introduction and overview

- To consider the nature and relevance of the topics of sequences and series.
- To discuss the relationship between mathematics and mathematics subject didactics.
- To introduce the theories and development levels of Piaget, van Hiele and Freudenthal.
- To give a brief outline of visualisation as well as patterning and generalisation.
- To motivate the relevance of thinking levels in the topic of sequences and series and the influence of visualisation, patterning and generalisation on advancement through these levels.
- To present the problem statement, aims, objectives as well as the content for each chapter, the analysis of the problem and research design.

Chapter 2 Historical survey

- To make a brief study of the historical background and nature of sequences and series in terms of the relevance of the study.
- To motivate why this topic should be studied by matriculation students.

Chapter 3 Mathematics educational aspects

- To make a study of the learning levels described by Piaget, van Hiele and Freudenthal.
- To make a brief study of visualisation, patterning and generalisation.
- To give an indication of the relevance of thinking levels in the development of concepts in sequences and series and the part played by visualisation, patterning and generalisation in learning mathematical topics.

Chapter 4 Empirical investigation

- To introduce a group of six students to sequences and series by using various visual illustrations, patterns, basic everyday examples and include some historical background.
- To consider Land's adaptation of van Hiele's learning levels in both the teaching approach and the preparation of work sheets and provide opportunities for visualisation, patterning and generalisation in the lessons.
- To include proof by mathematical induction as a fourth level activity to enhance and help to ascertain the students' attainment of the fourth level.
- To consider the characteristics of the qualitative and quantitative research approaches.

Chapter 5 Presentation and interpretation of data

- To draw up a series of questionnaires to determine pupils' achievement in their thinking levels and establish the significance of visualisation, patterning and generalisation in the development of pupils' thinking levels in terms of the four levels described by Land (1990).
- To observe the students' responses to the teaching approach.
- To draw up graphs to illustrate the results obtained in the questionnaires as related to the influence of visualisation, exploring patterns and generalisation on the advancement from level 0 to 1, 1 to 2 and 2 to 3.
- To analyse student responses to the questionnaires to obtain an indication of the

advancement of their thinking levels and the part played by visualisation, patterning and generalising in this regard.

Chapter 6 Conclusion and recommendations

- To draw up conclusions from the results of the investigation.
- To make recommendations based on the findings.

1.4 Analysis of the problem.

Brown refers to mathematics as "... the science, craft and art of pattern and structure" (Sierpiska & Kilpatrick 1998: 462) which, as such, underlies so many other scientific and technical areas involved in searching for patterns and structure to be found in nature. The recognition of patterns often leads to the establishment of rules and theorems. In order for this to be meaningful to matriculation learners studying sequences and series, they need not only to be equipped with some basic mathematical knowledge but also have their thought processes elevated to enable them to reason on a more abstract level. Steen refers to "... the steep and harsh terrain of abstract language that separates the mathematical rain forest from the domain of ordinary human activity" (Grouws 1992: 39).

As the students advance to higher levels in mathematics, they need to be able to engage in a more abstract type of exploration of the structure of mathematics. Freudenthal states that: "... Mathematics tells us nothing about reality but that one thing follows from another in a certain way" (Freudenthal 1967: 6). The approach adopted here will not be the traditional one which was to begin lessons by giving facts and steps in an algorithm. The pupils were given no opportunity to discover and perform relevant results and applications in mathematics. Instead the teacher did model examples from the textbook and set exercises on the work for their pupils. Although there is plenty of evidence in literature supporting the problem-solving approach of how children learn and should be taught mathematics, even in today's secondary schools: "... Emphasis is almost entirely on algorithmic skills. Neither understanding of why rules and algorithms work nor applications to real-life problems are stressed" (Cangelosi 1996: 29). In subsequent lessons students will be provided with a

variety of pattern structures and given the opportunity of establishing rules for themselves.

Freudenthal notes that "... didactics is about the arrangement of the context of mathematics instruction, broadly speaking, but it is also the science of structuring" (Streefland 1993: 109). Human beings have the desire to organise the patterns they find and this leads to advancement to higher levels of thought. Here students will be encouraged to generalise and find out and prove why logical structures work and thereby reach the theoretical level. They will discuss their thoughts, work together and learn to employ both the standard symbols and language used in a particular field.

It seems that visualisation may serve a valuable purpose in pre-instructional activities and the retrieval phase of the learning process referred to by Dwyer (1988: 365). However, Arnitsur comments that, although that he is in favour of making use of this additional sense of seeing,

... being able to discover mathematical facts by the help of graphs is definitely not enough. One has to be able to prove the facts in only mathematical ways. He must be able to draw conclusions (Sierpiska & Kilpatrick 1998: 493).

Nevertheless, since visualisation does play a valuable role in introducing and promoting understanding of topics, in the teaching-learning situation, "... visually-orientated activities are being introduced into mathematics classrooms and their place in the curriculum is steadily growing" (Sierpiska & Kilpatrick 1998: 492). The lessons on sequences and series will involve the use of suitable visual representations to complement instruction while attempting to ensure that students actually do relate visual illustrations to the underlying concepts.

As a result of the traditional teaching approach, learners in the secondary phase in South Africa tend to experience problems in the mastering of formulae and procedures in the topic of sequences and series. Patterning in sequences and series is not generally seen as being related to visualisation as well as spatial thinking and reasoning. Since the processes of patterning and generalising are not emphasised, students experience difficulty with spatial

patterns. Generalisations and algorithms become meaningful when their origins are understood. Further, the fact that their problem solving skills and strategies are not well developed, contributes to their limited success in solving word problems based on sequences and series.

Hargreaves, Threfall, Frobisher and Sharrocks-Taylor have conducted research on the "... understanding and performance of primary school children in patterns prescribed as a sequence of symbols" (Orton 1999: 67). The studies suggest that children of all ages should be provided with a much wider array of pattern structures and tasks within the classroom. They have found that the provision of a large range of pattern structures encourages children to employ a variety of strategies to make a generalisation and be more persistent in their search for patterns. The provision of patterns helps to enhance their grasp of concepts and principles by gradually improving their ability to analyse physical phenomena and make predictions in new or altered situations. The implications of these findings will be taken into consideration here.

Once patterns have been established, the goal is to move beyond to a further level involving generalising the findings made. Generalisation refers to going beyond what is explicitly given. In order to promote better strategy use and improved generalising, pupils will be provided with sequences covering a wide range of structures. Since patterns and making of generalisations are very important in the study of mathematics, the development of generalisations regarding sequences and series helps to establish a necessary mathematical foundation for the future.

Proof forms part of the fourth of van Hiele's intellectual levels, the theoretical level. In the lessons on sequences and series here, the provision of rich schematisation should lead to the development of the student's desire to understand logical connections in the work being studied. Movshovitz - Hadar (1993: 256) observes that "... there seems to be a general consensus that mathematical induction should be introduced in high school".

The NCTM's Curriculum and Evaluation Standards (1989) expressed the need for more attention to be paid to proof by mathematical induction because, although this is a very important tool to mathematicians, it has always been a difficult topic for students to master. In order to promote the understanding of proof, in subsequent lessons, initially numerous examples of apparent number patterns and generalisations will be provided. These will be followed by visual representations leading up to the formal mode of proof. Students would be encouraged to understand when this would be applicable. A desirable goal for teaching some proof in these lessons would be to assist matriculation pupils with their current mathematics and prepare them for possible future studies in the subject.

Although only arithmetic and geometric sequences are required to be studied at school, the Fibonacci sequence is an interesting type which occurs frequently in nature and so is covered here as well. Examples given serve to stimulate the pupils' interest and increase their understanding of the concept of a sequence. They would be able to appreciate how old a concept it is by being given the historical background of Leonardo Fibonacci, who was born in 1170 AD and died in 1250 AD. Some of the applications to infinite series and proof by mathematical induction are also not on their syllabus but are included as enrichment and to advance the students' mathematical thinking. The developmental phases identified by the van Hiele are taken into consideration and the aim is to take the learners through the various thought levels so that, as a result of being provided with visual representations and patterns, they are able to reinvent for themselves the relevant theory regarding sequences and series.

1.5 Research design.

The research design is comprised of: a literature research regarding the historical value and relevance of sequences and series; an investigation of the theories of Piaget, van Hiele and Freudenthal as well as the importance of visualisation, patterning and generalisation in the formation of concepts; an empirical experiment to determine the significance of the learning

levels in the development of the concepts of sequences and series and the influence of the inclusion of activities emphasising visualisation, patterning and generalisation. The experiment involves a series of lessons in which learning levels and the influence of visualisation, patterning and generalisation are considered combined with a series of seven intensive questionnaires to determine pupils' achievement in their thinking levels and to give an indication of the relevance of visualisation, patterning and generalisation in the development of their thinking processes. Thereafter graphs drawn up to illustrate the ratings attached to answers on questionnaires and designed to illustrate the influence of visualisation, exploring patterns and generalisation in the rise from levels 0 to 1, 1 to 2, 2 to 3 and in general will be presented. This will be followed by an analysis of findings emerging from the experiment by means of graphical representations as well as the drawing of conclusions and making of recommendations.

The investigation was done with six grade twelve students, including three boys and three girls, who were medium and top performers in their classes and were students of one of the English secondary schools in Centurion. They were exposed to fourteen one and a half hour sessions, working in two small groups of three each. The material was prepared on worksheets and the students had to complete seven intensive questionnaires on the processes completed and the evaluation of their progress. The introductory lessons were designed to create learning experiences and opportunities for pupils and to investigate their progress through thinking levels while they were being provided with opportunities of visualisation, patterning and generalisation.

1.6 Layout of the dissertation.

Chapter 1: Introduction and overview

In this chapter the motivation for this dissertation will be given, the problem statement presented, aims and objectives listed, the problem analysed, research design discussed and layout outlined.

Chapter 2: Historical survey

In this chapter the historical background and the relevance of the topic will be considered.

Chapter 3: The learning theories of Piaget, van Hiele and Freudenthal and the importance of visualisation, exploring patterns and generalisation in mathematics

The learning theories of Piaget, van Hiele and Freudenthal will be investigated here and their implication for learning levels in the topic of sequences and series. The impact of visualisation, exploring patterns and generalisation on advancement through learning levels will form a significant part of the study.

Chapter 4: Applications of theory to the learning of sequences and series

Learning levels involved in learning sequences and series based on Land's adaptation of van Hiele's learning levels to teach functions will be identified. Extracts from appropriate lessons involving visualisation, exploring patterns and generalisation will be provided. The significance of qualitative and quantitative methods of research will be discussed.

Chapter 5: Research / data analysis

The responses to seven questionnaires of the six medium and high ability matriculation students as they are taught the lessons in chapter 5 will be given and analysed in various categories. Graphs will be provided to give an indication of the progress through the various levels. The fact these questionnaires are detailed and will be provided over frequent and regular intervals implies that they should provide comprehensive coverage of the overall progress of the students.

Chapter 6: Conclusion and recommendations

Conclusions will be made from the analysis of student responses and recommendations made for future application and research.

CHAPTER 2

Historical survey

2.1 Introduction

In this Chapter the importance of the roles played by the history of mathematics, particularly the history of sequences and series will be investigated. Consideration will be given to the relevance of the topic in history, in mathematics and in the lives of matriculation students. Sequences and series involves patterns and history reveals the significance of patterns in mathematics. Lynn Arthur Steen observes:

Mathematical theories explain the relationships among patterns; applications of mathematics use patterns to 'explain' and predict natural phenomena that fit these patterns. Patterns suggest other patterns, often yielding patterns of patterns. In this way mathematics follows its own logic, beginning with patterns from science and deriving new patterns from the initial ones (NCTM 1989 : 487).

2.2 The relevance of the history of mathematics

Before writing was invented, mankind used to make calculations and consider geometric figures. Numbers appeared together with the first writing and soon after that great developments took place in mathematics. From the beginning mathematics has been used for many practical requirements including commerce, tax, calendars and land surveying. However, mathematics has always served an even greater purpose than this. Freudenthal states that " This game of numbers and figures was an end in itself" (Freudenthal 1967: 6).

Pure mathematics developed well in form and context and the Greeks added a logical system to what they inherited from the Babylonians. As a result, ever since then, mathematics often starts with certain fundamental assumptions and draws conclusions by means of logical deductions called proofs. Other sciences gradually began to make more and more use of mathematics. It was astronomy that first began to do this and since then the possibilities of applying mathematics in different fields have continually expanded. As mathematics has developed, it has in the words of Freudenthal (1967: 9) "... expanded not only beyond its own frontiers but also across the boundaries separating the different parts of

mathematics".

The historical development of mathematics reveals successive generations developing their own perceptions on mathematical ideas, based on mutual agreement they have reached regarding important concepts. The French chemist Antoine Laivoisier describes how "... the human mind gets creased into a way of seeing things" (Grouws 1992: 495). History helps to show the nature, role and meaning of mathematics and reveals the fascination mathematics has held for many successive generations. In addition,

... History can do more than present stimulating problems and ideas, it can also help students to perceive relationships and structure in what appears to be a tangled web of geometry, algebra, number theory, functions, finite differences, and empirical formulas (NCTM 1989b: 16).

Furthermore, an awareness of the history of mathematics helps to stimulate the interest and appreciation of the pupils. The following quotation by Jacques Barzun, an American teacher, reflects problems that may arise in the algebra teaching / learning situation:

... I have more than an impression - it amounts to a certainty - that algebra is made repellent by the unwillingness or inability of teachers to explain why There is no sense of history behind the teaching, so the feeling is given that the whole system dropped down ready-made from the skies, to be used only by born jugglers (NCTM 1989b: 1).

Phillip Jones gives certain categories of "whys" including the "chronological whys" and "logical whys" (NCTM 1989b : 2) in the significance of the history of mathematics. The chronological whys show the development of definitions and systems and how new systems are developed by "... being generalised and rephrased to include the new ideas as well as the old" (NCTM 1989b: 2). The logical whys involve an understanding of the nature of the axiomatic systems, logical reasoning and proofs. Phillip Jones notes that

... It is important that our students grow to understand this structure, but for many topics the direct and minimal statement of axioms and proofs is neither the way these ideas developed historically nor the way perceptions grow in the minds of many of our students (NCTM 1989b: 2).

This reflects Freudenthal's idea of "... the genesis in reverse as a didactical principle" (Streefland 1993: 122) and that mathematics should be taught as a reinvention. Phillip Jones quotes the authors of the article "On the Mathematics Curriculum of the High School" describing "the genetic method" which suggests as a general principle to let a person

"... retrace the mental development of the race-retrace its great lines, and not the thousand errors of detail" (NCTM 1989b: 3).

Phillip Jones points out the connections to be found in history between mathematics and philosophy, mathematics and religion, mathematics and music and art and, further, how "... practical, social, economic, physical needs often serve as stimuli to the development of mathematical ideas" (NCTM 1989b: 8). However, he stresses that although mathematics may be stimulated by physical problems or geometric problems, the mathematics created is an abstraction. For, "... often the applications of a mathematical concept in a system are unforeseen by the inventors and may follow years later in unpredicted ways" (NCTM 1989: 11). Durbin (1973) comments on how mathematicians are not always drawn to problems merely for practical reasons and yet later their findings have led to very practical consequences. James Phillips observes how

... the generalisations and abstractions of mathematics are really the useful parts because it is they that make the mathematics applicable to physical situations as yet unknown and perhaps even unforeseen. Students benefit by being taught the history of generalisation in mathematics so that they are able to appreciate the nature and role played by generalisation, axiomatisation and proof in the subject. (NCTM 1989b: 13).

Generalisation and proof form a significant part of the lessons in Chapter 4. As students move towards more advanced mathematical thinking, they need to make a difficult transition from a position where concepts are formed on an intuitive basis from their own experience to formal definitions and deductions. The research done here has been designed to help them bridge that gap effectively and to encourage them to find enjoyment in solving mathematical problems. Durbin (1973) aptly speaks of a special spirit of mathematics which has led to its evolution and needs to be conveyed to students of the subject.

2.3 Historical background of sequences and series

There is abundant evidence of pattern and order all over the world: Hence, the sequence, which is an ordered list of terms, plays a significant role in daily human activity. In everyday language the word sequence is often used. For example, in history a teacher may speak of

certain events following one another in a specific order represented by E_1, E_2, E_3, E_4 as follows:

E_1 : Versailles Treaty

E_2 : World depression

E_3 : Hitler's ascendancy

E_4 : Munich agreement (Fleming & Varberg 1989: 410).

Here E_1, E_2, E_3, E_4 forms a sequence of events and similar notation is used in the case of number sequences.

Originally the word "progression" from the Latin word "progressio" was used for a sequence by Boethius (in approximately 510) and other Latin writers. This was replaced by the modern term of sequence by the end of the seventeenth century. In ancient times writers used to connect progression with proportion, using the names of arithmetic, geometric and harmonic to describe each type. In early books a proportion is called a progression consisting of four terms. In addition, "[ek' thesis], literally a setting out" (Smith 1953: 496) is the Greek name for a series used by the early Pythagoreans and the name given to a term of a series was "ὅ pos (hor'os)", literally "a boundary" (Smith 1953: 496). The name series seems to have originated from the seventeenth century. For example, James Gregory in 1671 mentioned "infinite serieses" (Smith 1953: 497) rather than series as it came to be known at a later stage. In this and subsequent chapters a sequence is referred to as being "... any ordered list T_1, T_2, T_3, T_4 " (Laridon 1996: 47) and a series as the indicated sum of the terms of a sequence.

Freudenthal has remarked that "The number sequence is the foundation-stone of mathematics, historically, genetically, and systematically. Without the number sequence there is no mathematics" (Freudenthal 1973: 171, 172). Although there are infinitely many types of sequences, only a few have been studied in great detail. It was the arithmetic and geometric sequences which first attracted attention and later, due to the Greeks, harmonic ones. (A harmonic series is a p -series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p equals 1). These three

were the only chief kinds studied by the ancients. Boethius, however, (of approximately 510) writes that there were an additional three types which were not given any specific names:

"... vocantur autem quarta: quinta: vel sexta" (Smith 1953: 494).

At times mention is made of special kinds of sequences such as the "astronomical progression" (Smith 1953: 495) $(1, \frac{1}{60}, \frac{1}{3600}, \dots)$ which is one of the few examples of decreasing sequences to be found in early European books. There were also Hindu, Arabian and Jewish writers who gave attention to the sums of the sequences of squares and cubes. Though series were generally considered as being ascending by medieval writers, long before the time Ahmes, Archimedes and certain Chinese writers had considered descending sequences. Before the 17th century, rather than talking about arithmetic, geometric and harmonic series, people spoke of natural, non-natural, continuous and discontinuous ones. For example, the sequence $(1, 2, 3, \dots)$ was regarded as a natural one while a discontinuous one was one in which the difference between consecutive terms was not unity.

In early books not much attention was given to series as they were regarded as being "... one of the fundamental operations" (Smith 1953: 497). Very little was written about them in the sixteenth century and in 1572 Digges gave only two pages to the topic. Most early writers limited their work to finding the sum of series but a few did give a rule for finding the last term of an arithmetic or geometric series without making any attempt to justify the rule. It was only in the 17th century that sequences and series were studied in much detail. This was done through a better algebraic system. Different cases were discussed and this led to the development of rules which have relevance in both pure mathematics and its applications.

At the beginning of the seventeenth century there appeared an "... interest in the infinitesimal as an element in the analysis" (Smith 1953: 506). At this stage the study of series having an infinite number of terms, which was already known to the Greeks, was revived and infinite products were also studied. One of the first of these products, expressed in modern terms, was:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{1 + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \dots \dots \text{(Smith 1953:506)}.$$

When differential calculus was discovered in the seventeenth century, numerous infinite series and continuous fractions for π were found. Progressively through history trigonometric functions could be separated from their dependence on circles, triangles and ratios. They become generalised and abstracted so as to be sets of ordered pairs of numbers which could be matched with one another by summing infinite series. This made them become more useful and Alfred North Whitehead remarked "Thus trigonometry became completely abstract, and in thus becoming abstract, it became useful" (NCTM 1989b: 13).

Sequences and series have been studied by many mathematicians over the ages.

Numerous patterns are evident in their formation. Both mathematicians and educationists are aware of the importance of pattern in mathematics. Sawyer (1955: 12) stated that "Mathematics is the classification and study of all possible patterns" (Orton 1999: vii) and Lynn Arthur Steen claims that "Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination" (NCTM 1989b: 487).

Since patterning is abundantly evident in sequences and series, the study of sequences and series is relevant for students as it gives them the opportunity to search for patterns in space, numbers and algebraic or geometric structures and then establish theories based on their findings. Sequences and series are suggested, for example, by spatial and number patterns, plant growth, financial matters, training programmes, computers and ordering of events, all of which form part of daily life. The topic is therefore not only meaningful to them but also encourages them to generalise and form abstractions so that they are able to advance to higher levels in their mathematical thinking. Establishing rules and proving that they are true would help students advance to the theoretical level of Van Hiele's thought levels. It is important for matriculation mathematics students to reach this level because, as Welder

states, modern mathematics

... is seen as a description of an external world of reality, nor merely as a tool for studying such a world, but rather a science in its own right. Having moved from the earlier dependence on natural phenomena for inspiration of new concepts, mathematics now finds most of its stimulus for new ideas from within itself (NCTM 1989b 461).

2.4 Arithmetic sequences and series

2.4.1 Introduction

The type of patterning found in arithmetic sequences is one of constant increase or decrease. It is evident in many everyday situations. For example, it is used in numbering of buildings, time periods, financial matters, architecture, designs and many other situations involving constant increase or decrease. The generalisation of this type of pattern led to the development of the concept of an arithmetic sequence over three and a half thousand years ago.

An arithmetic sequence may be defined as "... a sequence in which the difference between any two successive terms is some constant d " (Laridon 1996: 50) and an arithmetic series as the indicated sum of the terms of an arithmetic sequence. Here the first term will be presented by the letter " a ". The general term is usually represented by T_k or T_n , n generally refers to the number of terms and S_n represents the sum of n terms of the sequence.

2.4.2 Arithmetic sequences

It was Ahmes Papyrus who, as early as 1550 BC, proposed two problems involving arithmetic sequences. The first problem and solution follows below:

"Divide 100 loaves among five persons in such a way that the number of loaves which the first two receive shall be equal to one seventh of the number that the last three receive".

The solution shows that an arithmetic progression is understood, in which $n = 5$,

$$S_5 = 100 \text{ and } \frac{(a + 4d) + (a + 3d) + (a + 2d)}{7} = (a + d) + a$$

Then, by modern methods $2d = 11a$.

$$\text{Therefore } 100 = \frac{2a + 4d}{2} \cdot 5 = 60a,$$

Whence $a = 1\frac{2}{3}$ and $d = 9\frac{1}{6}$.

Therefore the series is $1\frac{2}{3}, 10\frac{5}{6}, 20, 29\frac{1}{6}, 38\frac{1}{3}$ although the method here given is not the one followed by Ahmes. (Smith 1953: 498).

The second problem, with its solution as given by Ahmes, reads as follows:

Rule of distributing the difference. If it is said to thee, corn measure 10, among 10 persons, the difference of each person in corn measure is $\frac{1}{8}$.

Take the mean of the measures, namely, $\frac{1}{16}$.

Take this 9 times.

This gives to thee $\frac{1}{2} \frac{1}{16}$.

Add to it the portion of the mean.

Then subtract the difference $\frac{1}{2}$ from each portion, (this is the order) to reach the conclusion.

Make as shown:

$$1\frac{1}{2} \frac{1}{16}, 1\frac{1}{4} \frac{1}{8} \frac{1}{16}, 1\frac{1}{4} \frac{1}{16}, 1\frac{1}{8} \frac{1}{16}, 1\frac{1}{16},$$

$$\frac{1}{2} \frac{1}{4} \frac{1}{8} \frac{1}{16}, \frac{1}{2} \frac{1}{4} \frac{1}{16}, \frac{1}{2} \frac{1}{8} \frac{1}{16}, \frac{1}{2}, \frac{1}{16}, \frac{1}{4} \frac{1}{8} \frac{1}{16}$$

This may be stated in modern form as follows:

Required to divide 10 measures among 10 persons so that each person shall have $\frac{1}{8}$ less than the preceding one. That is, $n = 10, S_n = 10, d = -\frac{1}{8}$, so that

$$S_n = 10 = \frac{2a + (n-1)d}{2} \cdot n = (2a - \frac{9}{8}) \cdot 5,$$

whence $a = 1\frac{9}{16}$, and the series is the descending progression.

$$1\frac{9}{16}, 1\frac{7}{16}, 1\frac{5}{16}, \dots, \frac{9}{16}, \frac{7}{16} \text{ (Smith 1953: 499)}$$

2.4.3 Arithmetic series

At the beginning of the Christian era the following problem (concerning the summation of terms of arithmetic sequences) was presented in a Chinese work: "There is a woman who weaves 5 feet the first day, her weaving diminishing day after day until the last day, she weaves 1 foot. If she has worked 30 days, how much has she woven in all? (Smith 1953: 499).

The following rule was proposed by an unknown author: "Add the amounts woven on the

first and last days, take half the sum, then multiply by the number of days" (Smith 1925: 499). In Europe naturally the same rule (but making allowances for language differences) came to be used. Sometimes this rule was put into verse so that it could be easily memorised. For example, the following Latin verse by (Huswirt, 1501) gives the rule

Si primus numerus cum postremo faciat par

Eius per mediu loca singula multiplicabis

Asi impar medium vult multiplicari locorum (Smith 1953: 500).

One of the greatest mathematicians of all times, Carl Friederich Gauss, who lived in the 19th Century, as a young schoolboy managed to add up the first hundred numbers quickly by adding them in pairs from the ends inwards as follows:

$$1 + 100 = 101$$

$$2 + 99 = 101$$

$$3 + 98 = 101$$

•

$$50 + 51 = 101$$

$$\text{Thus } 1 + 2 + 3 + \dots + 100 = \frac{100 \times 101}{2} = 5050$$

Here the pattern indicates that the aforementioned rule of adding the first and last numbers, halving the sum and multiplying by 100 was employed. By means of this method, a formula

$$S_n = \frac{n}{2} [2a + (n - 1)d]$$

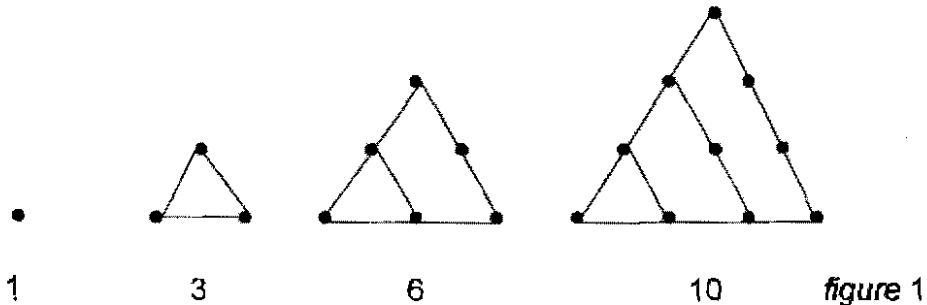
may be derived for the sum of the first n terms of an arithmetic sequence. Using the Greek letter \sum to denote summation and $a + (k - 1)d$ to represent the

k th term, we may write the sum as $\sum_{k=1}^n [a + (k - 1)d]$ which equals $\frac{n}{2} [2a + (n - 1)d]$.

2.4.4 Figurate numbers and arithmetic means

The ancient Greeks were aware of the theory of arithmetic series. However, they usually associated it with polygonal numbers. It was natural for the early Pythagorians to consider geometrical or physical representations of points in a plane and study their resulting properties. The Greeks showed a deep interest in numbers connected to geometric forms, which received the name of figurate numbers. Figurate numbers, which are regarded as the

number of dots in certain geometrical configurations, represent a link between algebra and geometry. Arithmetic series are reflected in the pattern arising from the following example:



Here the n th triangular number, T_n , is given by the sum of an arithmetic series which is

$$1; 1 + 2; 1 + 2 + 3; 1 + 2 + 3 + 4; \dots$$

Diophantus was an ancient Greek mathematician who studied the properties of the figurate rational numbers. He did not prove any general theorems about figurate numbers but he introduced the technique of writing down equations with a symbol for the unknown number. Pappus was another outstanding mathematician of this period. Around the year 320, he wrote an original work known as the Collection (Synagoge) which consisted of eight books. In Book III he discusses the problem of the two mean proportional and places it in the context of the general theory of arithmetic, geometric and harmonic means. The arithmetic mean holds a significant place in history. C. Neumann (1870) invented the 'method of the arithmetic mean' to establish the existence of a harmonic function ϕ which takes a prescribed value m on a convex surface σ (Temple 1981: 236).

2.4.5 Conclusion

Evidently arithmetic sequences and series do go very far back in history. Even in about 1100BC the odd and even number sequences appear in an ancient Chinese book.

Smith (1953) claims that the ancient Greeks had the superstition that the odd numbers (except thirteen) were fortunate and masculine, divine and heavenly while the even numbers were unfortunate, feminine, human and earthly. They were fascinated by the visual representations of sequences and so this would seem to be a relevant starting point for mathematics students today. As has been mentioned, arithmetic sequences and series are

evident in many situations such as in patterns of numbering of houses along a street, reading of scales, economic situations, training programmes, leap years, days of a week and many other instances too. Finding the pattern and establishing a rule in these situations helps students rise to van Hiele's second level of thought. Working with formulae and proving results raises them to the third and fourth levels of thinking. The topic of arithmetic sequences and series is relevant to matriculation students since it can promote the development of their mathematical thinking and can be taught as a reinvention as advocated by Freudenthal.

2.5 Geometric sequences and series

2.5.1 Introduction

The type of pattern evident in geometric sequences result from constant multiplication by the same number. The fact that these sequences can be seen in family trees, financial matters, growth, designs and many other situations besides, led to the generalisation of the concept of a geometric sequence over four thousand years ago. A geometric sequence may be defined as "... one in which the ratio of any two successive terms is a constant" (Laridon 1996: 54) and a geometric series as the indicated sum of the terms of a geometric sequence. Here " a " will represent the first term and " r " the common ratio. The general term is usually represented by T_k or T_n , n usually refers to the number of terms of a finite sequence and S_n represents the sum to n terms of a sequence.

2.5.2 Geometric sequences

Benjamin Franklin in his "Poor Richard's Almanack" for the year 1751, calculated the number of a person's ancestors 30 generations back. His list began like this:

- | | | |
|----|--|----|
| 1. | A present man's Father and Mother were | 2 |
| | His Grandfathers and Grandmothers | 4 |
| | His Great Grandfathers and Great Grandmothers | 8 |
| | and, supposing no Intermarriages among Relations, the next predecessors will increase to | 16 |
- (Jacobs 1970: 50).

Here 2, 4, 8, 16, is the number sequence for the relatives. Each subsequent term of the sequence is found by multiplying by 2.

There is a problem about Jacob's golden staircase which involves geometric sequences. It reads as follows:

"In his dream, Jacob saw a golden staircase with angels walking up and down. The first step was 8 inches high but after that each step was $\frac{5}{4}$ as high above the ground as the previous one. How high above the ground was the 800th step?" (Fleming & Varberg 1989: 419).

Calculating the height of the height of the 800th step, it comes to $8(\frac{5}{4})^{799}$ which is approximately $3,4 \times 10^{73}$ miles high as opposed to the distance from the earth to the sun which is approximately $9,3 \times 10^6$ miles. This would certainly suggest that this would be a staircase for angels and not people.

2.5.3 Geometric series

The first cases of geometric series ever found are those of the Babylonians in c2000BC. The earliest example found in Egyptian mathematics belonged to Ahmes Papyrus in approximately 1550BC and is quoted below:

<i>The one Scale</i>		<i>Household</i>	7
Once gives	2 801	<i>Cats</i>	49
Twice gives	5 602	<i>Mice</i>	343
Four times gives	11 204	<i>Barley (Spelt)</i>	2 301 (sic)
Together	<u>19607</u>	<i>Hekt measures</i>	16 807
		<i>Together</i>	<u>19607</u> (Smith 1953: 500).

It seems that the intention is to deduce a rule for summing the terms of a geometric sequence in the left hand column.

As a result of observing patterns arising from summation, the Greeks had rules for summing series and Euclid gave one as follows:

$$\frac{a_{n+1} - a_1}{a_n + a_{n-1} + \dots + a_1} = \frac{a_2 - a_1}{a_1}$$

The Hindus were mainly interested in the summation problems of geometric series. The

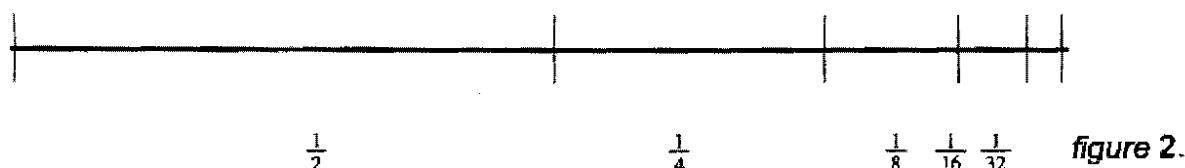
Arabs obtained the rule for summation from the Greeks and "... it appears in an interesting form in the chessboard problem in the works of Albêrûnî (c1000)* (Smith 1953: 500). The chessboard problem involves geometric sequences. The king of Persia was so enchanted when he learnt to play the game of chess, that he promised to reward the inventor Sessa by fulfilling any request he might make. Sessa wished for one grain of corn to be placed in the first square, two on the second, four on the third and so on and the king agreed. However, the total number of grains would be $1 + 2 + 2^2 + \dots + 2^{63}$ which was clearly an impossible request to fulfill.

Apparently medieval writers obtained their rules from the Arabs as it appears in the "Abaci" of Fibonacci in 1202. In the "Algorithmus de Integris" (1410), Prosdocimo de' Beldamandi gave the rule in the form $a + ar + ar^2 + \dots + ar^{n-1} = ar^{n-1} + \frac{ar^{n-1} - a}{r - 1}$. According to Smith (1953), later Chuquet (1484) gave the rule in the form $S = \frac{rar^{n-1} - a}{r - 1}$ and the present rule of $S_n = \frac{a(r^n - 1)}{r - 1}$ was given by Clarrus (1583).

2.5.4 Summing infinite geometric series

Since Zeno first introduced his famous paradoxes of the infinite over 2000 years ago, great thinkers have been intrigued by questions such as whether or not sums such as

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \dots$ and $1 + 3 + 9 + 27 + \dots$ make sense. In the former case, substitution of the first term $a = \frac{1}{2}$ and the common ratio $r = \frac{1}{2}$ into the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ gives $1 - (\frac{1}{2})^n$, which tends to 1 as n tends to infinity because $(\frac{1}{2})^n$ tends to zero as n tends to infinity. This result may be visualised by considering a piece of string cut into infinitely pieces as follows,



It would appear that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$

The first infinite series ever known to be summed was

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots + \left(\frac{1}{4}\right)^n + \dots$$

by Archimedes (in approximately 225 BC) in his quadrature of the parabola. Vieta (in approximately 1590) gave the general formula $S_{\infty} = \frac{a}{1-r}$ for summing the infinite series

$$a + ar + ar^2 + \dots \text{ where } r < 1.$$

(If r is allowed to be negative the restriction would be $-1 < r < 1$)

On the basis of the geometric series $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$, Euler concluded that

$$1 - 5 + 25 - 125 + \dots = \frac{1}{1+5} = \frac{1}{6}$$

2.5.5 Conclusion

Michael Stifel (1486 - 1567) was one of the first German algebraists of the sixteenth century and his best known mathematical work is the "Arithmetic integra", published in 1544 and divided into three parts. Eves (1983: 199) notes how "... In the first part, Stifel points out the advantages of associating an arithmetic progression with a geometric one, thus foreshadowing the inventions of logarithms nearly a century later."

Evidently, ever since ancient times geometric sequences and series have attracted the attention of different nations. Their patterns arise in many contexts and students could see their application not only in various financial matters such as the calculation of compound interest but also in any other situation in which successive terms are calculated by repeated multiplication. Patterns and visualisation may be utilised to help establish the concept and then find and also prove rules for operating with geometric series.

2.6 Fibonacci sequences

2.6.1 Introduction

Fibonacci was one of the greatest mathematicians of the middle ages. He was born in Pisa in Italy and in 1202 he wrote a book on arithmetic and algebra. One of the problems proposed in his book was

... A pair of rabbits one month old are too young to produce more rabbits, but suppose that in the

second month and every month thereafter they produce a new pair. If each new pair of rabbits does the same, and none of the rabbits die, how many pairs of rabbits will there be at the beginning of each month? (Jacobs 1970: 72 & 73).

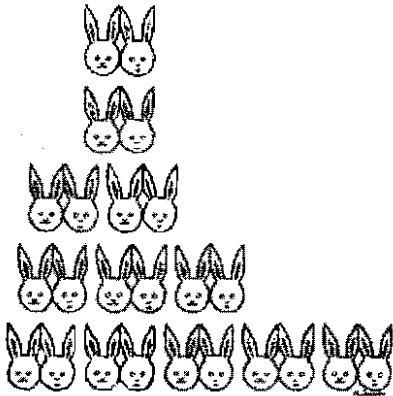


figure 3. (Jacobs 1970: 72).

The numbers that solve this problem form a pattern reflected in the sequence

1, 1, 2, 3, 5, 8, 13,

which is called the Fibonacci sequence.

The first two terms of the sequence are 1 and each succeeding term is the sum of the previous pair of terms.

$$\begin{aligned}
 1 &= 1 \\
 1 + 1 &= 2 \\
 1 + 2 &= 3 \\
 2 + 3 &= 5 \\
 3 + 5 &= 8 \\
 5 + 8 &= 13 \text{ etcetera.}
 \end{aligned}$$

If we denote the k th Fibonacci number by F_k , then we could define the Fibonacci sequence as follows

$$F_{k+1} = F_k + F_{k-1} \text{ for } k = 2, 3, 4, \dots$$

$$F_1 = F_2 = 1$$

2.6.2 Further examples of Fibonacci sequences

The pattern of the Fibonacci sequence appears in such seemingly unrelated topics as the family tree of a male bee and the keyboard of a piano.

"... A male bee has only one parent, his mother, while a female bee has both father and mother. The family tree

of a male bee has a strange pattern as a result. If each male is represented by the symbol ♂ and each female

by the symbol ϕ , the tree, illustrated below, forms a Fibonacci sequence. Further, the thirteen ancestors on the bottom of the tree reveal the same relationship as the keys of a piano keyboard as illustrated by the thirteen keys of one octave of the "chromatic" scale Jacobs (1970: 74).

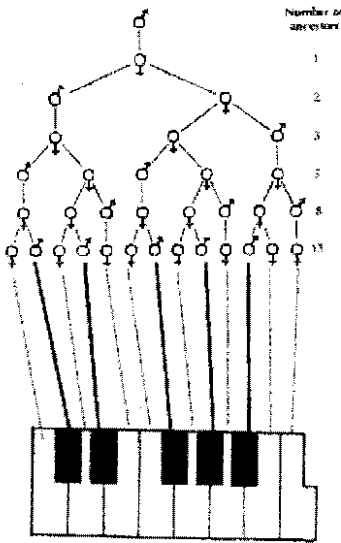


figure 4. (Jacobs 1970: 73 - 74)

Fibonacci numbers also appear in the pattern of the arrangement of leaves on the stems of plants, which will be studied by students in subsequent lessons. Fibonacci sequences even arise in the study of pineapples, where adjacent scales form spiral lines. As a result of the hexagonal form of the scales, there are generally three families of them:

- a family of x spirals going in one direction
- a family of y spirals which goes in the opposite direction and
- a family of z spirals going in the former direction, though in a steeper manner.

The spirals may be counted to determine the values of x , y , and z .

Jean & Johnson commented that

... In 1933 a worker in a society in Hawaii publishing the Pineapple Quarterly, noticed that x , y and z take the respective values 5, 8 and 13, and more often the values 8, 13 and 21 for bigger fruits. These are triples of consecutive Fibonacci numbers (Jean & Johnson 1989: 487).

2.6.3 The golden ratio

The Golden Ratio ϕ was very well known to the ancient Greeks. Here ϕ stands for Phidas who was the greatest sculptor of antiquity. ϕ is closely related to the Fibonacci numbers. The number ϕ is defined by $\phi = \frac{\sqrt{5} + 1}{2}$ and has a value of approximately 1,618 when expressed

in a decimal form. In order to see how the number ϕ is related to the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21,, consider the ratios of consecutive Fibonacci numbers

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \dots$. As the Fibonacci numbers become larger, their ratios approach the

golden ratio. In other words, this means that $\phi = \lim_{k \rightarrow \infty} \frac{F_{k+1}}{F_k}$. The ancient Greeks were

fascinated by the aesthetic proportion generated by ϕ in geometry, art and architecture and tests have revealed that rectangles for which the ratio of the base to the height is ϕ are most pleasing to the eye.

2.6.4 Conclusion

The Fibonacci sequence is such an interesting one which occurs frequently in nature. These examples reveal one of the fascinating aspects of mathematics that an idea that applies to one area of study frequently turns out to be valuable in another entirely unrelated area. In addition, not only do these examples illustrate the relevance of the concept of a sequence, but they reveal how old a concept it is since the pattern evident in a Fibonacci sequence was detected and generalised so many centuries ago.

2.7 Limits of terms of sequences and sums to infinity of convergent series

The ancient Roman poet Horace in his satires of 65 to 68 BC made the following statement about limits:

... est modus in rebus, sunt certi denique fines, Quos ultra citraque nequit consistere rectum.

[Things have their due measure, there are ultimately fixed limits, beyond which, or short of which, something must be wrong] (Grouws 1992: 501).

Ancient mathematicians were disturbed by the prospect of an infinite process. Whenever the Euclidean algorithm was used to find the greatest common divisor of two integers, the process came to an end because the set of positive integers has one as the smallest positive integer. However, when they tried to find the greatest common measure of two incommensurable line segments, the process went on forever. When Aristotle and other Greek philosophers of their time tried to answer the paradoxes of Zeno, their "... replies were so unconvincing that mathematicians of the time concluded that it was better to shun infinite processes altogether" (NCTM 1989b: 379). One of the earliest theorems on limits

(dating back to the time of the ancient Greek mathematicians) is given below in an awkward geometrical form (NCTM 1989b: 370) as follows:

"... for the crux of the matter is that if A is the greater of the two given (positive) magnitudes a and A , and if $u_n = \frac{A}{2^n}$, then $\lim_{n \rightarrow \infty} u_n = 0 < a$ " (NCTM 1989b: 379).

The word limit has many connotations in everyday life which do not have the same meaning as the mathematical limit. Very often an everyday limit means something that cannot and should not be passed such as a speed limit. In mathematics the words "tends to" and "approaches" are often used in connection with a limit and frequently it is not possible for the sequence to reach the limit. The notion of limit is surrounded by mystery and David Tall makes the following relevant observation:

Although the function concept is central to modern mathematics, it is the concept of a limit that signifies a move to a higher plane of mathematical thinking. As Cornu (1983) observed, this is the first mathematical concept that students meet where one does not find the result by a straightforward mathematical computation (Grouws 1992: 501).

Davis & Vinner believe that there are "... seemingly unavoidable misconception stages with the notion of a limit" (Grouws 1992: 501). These include the influence of language and the complexity of the idea which is unable to appear "... instantaneously in complete and mature form" with the result that some parts of the idea will get adequate representations before other parts will" (Grouws 1992: 501). In the lessons taught in chapter 4 students will be introduced to the idea of a limit too in order to find sums of convergent infinite series. They will not be given the formal definition of the limit of a series being the real number l such that given $\xi > 0$ there exists $M \in \mathbf{N}$ such that $n > M \Rightarrow |a_n - l| < \xi$ as they are certainly not ready for such rigour. However, visual representations such as computer values, graphical illustrations and block diagrams will all be used to help them develop the concept of a limit. Furthermore, they will consider various types of convergent sequences in order to give them a broader picture and an increased appreciation of the concept. For instance, it can be shown that the terms of the sequence $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \dots$ tend to $\sqrt{2}$ as the number of terms tends to infinity. The Golden Ratio provides another example in which the ratio of

consecutive terms of a sequence approaches a limit. These examples will be studied by students in Chapter 4 to enhance their understanding of the concept of a limit so that they appreciate the formula for the sum to infinity of a convergent geometric series. This would also be beneficial to those intending to study mathematics at university level.

2.8 Power series

2.8.1 Taylor and Maclaurin series

Gottfried Wilhelm Leibnitz (1647 - 1716) practised summing infinite series. He would start from a known sum and then derive the series from what he described as the "harmonic triangle" (Cooke 1997: 340). It has been said of Sir Isaac Newton (1642 - 1727) that "... There has never been a scientist like Newton, and there has not been one like him since." (Beckmann 1977: 134). He was born on Christmas day, 1642 in a small farm house at Woolsthorpe near Colsterworth, Lincolnshire. In the early years of calculus, it was a great triumph when Newton and others discovered that many known functions could be expressed as "polynomials of infinite order" or "power series" with coefficients formed by means of elegant laws. A power series is a series of the form

$c_0 + c_1(x - a) + c_2(x - a)^2 + \dots + c_n(x - a)^n + \dots$ where a and c_i are all constants.

Both Sir Isaac Newton and Leibnitz knew several power series expansions. Brook Taylor (1685 - 1731) and Johan Bernoulli were their disciples and they discovered the general procedure for generating representations of series in terms of derivatives of the functions represented. The series is now known as Taylor series. For example:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n \text{ is the Taylor series for a function } f \text{ about the point } a.$$

Colin Maclaurin (1698-1746), a contemporary of Taylor, was a brilliant Scottish mathematician, who as a result of a competitive examination, became a professor at the University of Aberdeen at the age of nineteen. He discovered the Maclaurin series which is a special case of the Taylor Series. For example, in the above Taylor series for a function f , for

the special case when $a = 0$, the Taylor series becomes $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} (x)^n$

and the right hand side is called the Maclaurin series for f .

Some interesting expansions include:

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$\frac{1}{1-x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$$

$$\begin{aligned} \frac{1}{x} &= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \\ &= (-1)^0 (x-1)^0 + (-1)^1 (x-1)^1 + (-1)^2 (x-1)^2 + (-1)^3 (x-1)^3 + \dots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots \end{aligned}$$

As has been mentioned in Chapter 1, the trigonometric expansions free the trigonometric ratios from definitions in forms of angles and ratio of side lengths of triangles. For example:

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} - \dots \end{aligned}$$

All of these above examples give an indication of the wonderful patterns to be found in power series and the concise way in which the pattern may be generalised by a formula.

2.8.2 The number e

The number e , called "Euler's number" (Reid 1992: 144) did not come into formal existence for nearly two thousand years after the Greeks began their numerical investigations.

Although the number e may not seem to be natural at all, it has a close connection to the natural numbers. The nearest exact numerical representation of e is the famous factorial series mentioned in 2.8 above:

$$e^x = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{x^n}{n!} + \dots \text{ (The Taylor expansion for } e^x \text{ with } x = 1).$$

This expansion will be dealt with by students in Chapter 4 to, introduce the concept of a limit.

The number e was found by Euler to be 2,71828182845904523536028 correct to 23 decimal places, a great feat for that time. It is the base of the mathematically natural logarithm, could be defined by $e = \lim_{h \rightarrow \infty} (1 + \frac{1}{h})^h$ and can also be expressed by infinite continued fractions as follows:.

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3 + \frac{3}{4 + \frac{4}{5 + \frac{5}{6 + \dots}}}}}} \quad \text{and} \quad e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \dots}}}}}} \quad (\text{Reid 1992: 155})$$

Striking patterns are evident in the above expansions for e . Taking the sums of large numbers of terms of these series enables students to appreciate the concept of convergence of series and to provide a basis for proof by mathematical induction.

2.8.3 The number π

Limits involving the irrational π will be included here too. During the course of history, different approaches have been followed to evaluate π .

For example, we have $\pi = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{f=0} \sqrt{n^2 - f^2}$ (Beckmann 1977:126)

or, using modern symbols, $\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2} dx$

Brouncker, using continued fractions, obtained the expression

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \dots}}}}} \quad (\text{Beckmann 1977:131})$$

Below is the first infinite series ever found for π (which was found from the Gregory series).

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$$

Leonard Euler was born in 1707 at Basel, Switzerland. He had a phenomenal memory and derived many formulae revealing beautiful number patterns for the squares of π , such as

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Furthermore, to calculate the logarithm of π , Euler found infinite products for powers of π

e.g.

$$\frac{\pi^2}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \dots$$

These are some of many infinite series found for π and the expansion

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \text{ will be investigated by students in Chapter 4.}$$

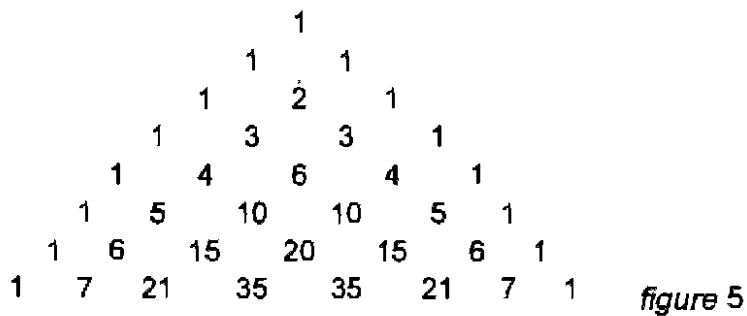
2.9 Number patterns

2.9.1 Introduction

As has already been mentioned in Chapter 1, Freudenthal felt that the reason why students often experience problems with mathematical induction is because "... people always began on far too high a level and do not give the pupil the opportunity to go through the previous, lower levels of the learning process" (Streefland 1993:37). The provision of number patterns and the opportunity of discovering apparent rules in the lessons in Chapter 4 are designed to overcome this problem. Some examples of number patterns, many of which will be studied by students in Chapter 4, will be mentioned here.

2.9.2 Pascal's triangle

Blaise Pascal (1623-1662) was a brilliant mathematician of the 17th century. He began his study of mathematics at age 12 and at age 13 he discovered the pyramid of numbers known as Pascal's triangle as illustrated below:



After beginning with a triangle of ones, a one is added to the beginning and end of each row. Each of the remaining numbers in the above triangle is the sum of the two numbers to the left and right above it. The r th row gives the coefficients of the terms of $(a + b)^r$, where $r \in \{0; 1; 2; 3; \dots\}$. The Binomial Theorem provides a method of expanding $(a + b)^n$ as a series in sigma notation as follows: $(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$. Cooke (1997:338) states that "it was the binomial series that really established the use of infinite series in analysis."

2.9.3 Configurate numbers

Reid (1992) notes how there has always been something solid about the symmetry of the number four. For example, each square is the summation of successive odd numbers and in this way the whole series of squares can be built up, layer by layer, from a single unit:

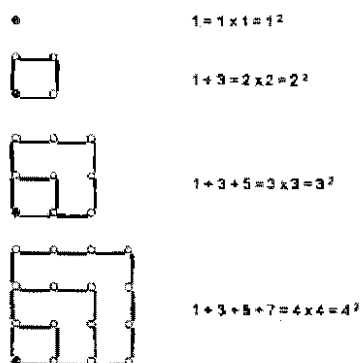
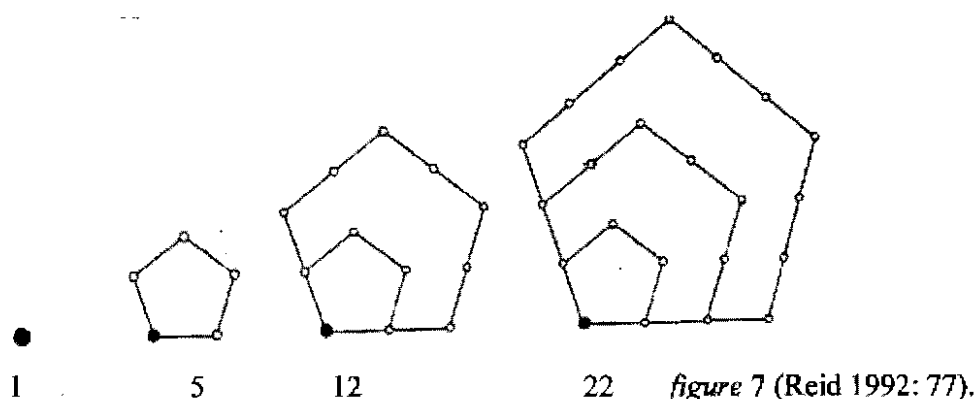


figure 6. (Reid 1992: 61)

Further, there are patterns to be found down the columns for even and odd squares too:

Even Squares	Odd Squares	All Squares
0	1	0
4	9	1
16	25	4
36	49	9

The Pythagorians were particularly interested in the shape of five and the pentagonal numbers. There is a pattern to be found in the subsequent pentagonal numbers:



An arithmetic series pattern is reflected as follows:

1; 1 + 4; 1 + 4 + 7; 1 + 4 + 7 + 10;

This example is in the lessons in Chapter 4 where the pattern is found, generalised and finally proved by mathematical induction.

Students will be asked to consider whether $n^2 - n + 41$ is a prime for every natural value of n . Since this is the case for the first forty natural values of n , it would certainly seem to be the case. However, $n^2 - n + 41 = 41^2 - 41 + 41 = (41)^2$ and so is not prime when $n = 41$. This shows that even though a statement may be true for the first forty values of n , it may not be true for $n = 41$. In this way the need for proof by mathematical induction becomes apparent.

As a result of correspondence with Michael De Villiers, the following two examples involving the divisibility of $2^n - 2$ and the verification of whether or not $S(n) = 991n^2 + 1$ is a perfect square are included here. In the fifth century BC Chinese mathematicians had already made the conjecture that if $2^n - 2$ is divisible by n then n must be a prime number. However, this conjecture was disproved only in 1819 when it was found that $2^{341} - 2$ is divisible by 341 although 341 is not prime since $341 = 11 \times 31$. The conjecture resulting from the Pell equation that $S(n) = 991n^2 + 1$ is a perfect square is true for all values of n until $n = 12\,055\,735\,790\,331\,359\,447\,442\,538\,767$. These examples certainly do appear to give further evidence of the necessity for proof of seemingly true statements.

There evidently are patterns which arise from sequences of numbers. Yet another example can be seen in the sum of the even natural numbers as follows:

$$\begin{aligned} 2 &= 2 = 1 \times 2 \\ 2 + 4 &= 6 = 2 \times 3 \\ 2 + 4 + 6 &= 12 = 3 \times 4 \end{aligned}$$

From the above, it would appear that the sum of terms of certain series may be represented by a formula.

2.9.4 Conclusion

Number patterns seems to lead to an awareness of the possibility of establishing a rule. For example, the previous example suggests that the sum of the first n even numbers is $n(n+1)$. However, a guess is not a proof and it is necessary to be able to prove that the result is true for an infinite number of cases. The type of proof required here is called proof by mathematical induction.

2.10 Proof

2.10.1 Introduction

Waring, Orton and Roper note that "The place of proof in mathematics teaching and learning has been a difficult, even contentious, issue of debate for many years" (Orton 1999 : 192). For a long time it has been Euclidean Geometry that has provided the main opportunity for introducing the idea of proof and logical argument in secondary school mathematics all around the world. The type of proof introduced by Euclidean Geometry is deductive. Although in many countries Euclidean geometry still forms the main systematic study of geometry in which deductive proof is taught in context, in Britain work with transformation geometry now forms a major component of the geometry curriculum although this does not lead to proof in the same way as Euclidean geometry does. However, since an education in mathematics demands the inclusion of proof in the curriculum, Orton observes that consideration needs to be given to the question: "To what extent can the approach to algebra

through pattern provide opportunities for learning something of the nature of proof in mathematics" (Orton 1999: 193).

Smith & Henderson state that:

A proof progresses from a stage of 'that which convinces' through a sequence of statements based upon the principles of reasoning (Stover 1989 : 16).

Judith Segal is concerned about what she terms

"...the distinction between the private and public aspects of a proof (conviction as compared with validity), and the fact that there are implicit non-logical criteria by which a proof may be judged as being acceptable to the mathematical community" (Segal 2000 : 182).

Hoffer (1981) expressed the belief that the high school course in geometry is often taught at a higher level than that attained by most students. Further, Stover (1989) notes how Senk and Usiskin began to question the importance of proof writing in high school geometry courses and that according to Ivanitsyna, students require not only a system of theoretical knowledge, but also general knowledge about a plan of reasoning. Theoretical knowledge is comprised of basic definitions and theorems while a plan of reasoning is made up of techniques of synthesis and analysis. Synthesis is an inductive process whereas analysis is a deductive process. The ability to deductively check a chain of events is one of the skills noted by Ireland in Stover (1989) to be involved in the construction of proofs. Further, Ireland claims that a knowledge of logical inference rules, a part of the deductive process, constitutes a necessary prerequisite for proof writing in geometry. The three proof strategies recognised by Carroll (1977) are

"... synthesis, analysis and combination methods" (Stover 1989: 3).

Michael de Villiers (1997) believes that in learning with or without dynamic tools, conviction is an important prerequisite for looking for proof. He refers to well-documented research that pupils do not perceive the need for proof (as a means of verification), especially in the case of visually obvious results. Consequently he observes that proof should become a more meaningful activity in the classroom with conviction being a prerequisite for proof rather than proof being a prerequisite for conviction.

2.10.2 Patterning and proof

Complaints have been made in the past by Hewit (1992) and Gardiner (1995) that studying patterns in mathematics "... goes little further than pattern spotting" Orton (1999: 193).

Orton (1999) notes how secondary school pupils might be able to spot and continue the pattern for the triangular number, only very few can find a formula for the general term and hardly anyone can ever prove it. He continues to raise what he terms unanswered questions such as:

Do pupils have the capability to create a proof or justification for generalizations from number patterns? Can the study of patterns be used in a deliberate way to introduce pupils to proof in mathematics? If so, what are appropriate ways to try to move pupils on from work with patterns to the idea of proof? Are pupils who have systematically and extensively studied number patterns and generalizations better at proof than pupils who have not? Orton (1999: 193)

These questions will be taken into consideration in subsequent chapters.

Orton carried out tests on teenage students in an attempt to find the effects of studying pattern in proof. Certain benefits did seem to arise from the experiment. Emphasis on pattern did seem to help pupils be aware of the need for proof and more receptive to proof.

However, it was not clear whether the ability to provide a reasonable proof was based on their natural ability, prior knowledge or understanding gained from the programme. It seems that progress towards finding a proof could be hampered by focus being misdirected from underlying structures. It was felt that encouraging students to use diagrams to analyse the structure of a pattern would be beneficial. However, the conclusion Orton drew was

...even within the fairly limited area of mathematics under consideration here, little transfer of competence in analysing patterns and providing proofs took place within the time scales available. Orton (1999: 208).

2.10.3 Mathematical induction

The method of proof by mathematical induction first became a part of algebra in the late sixteenth century. This method is evident in the proof of the infinitude of the number of primes by Euclid. Here it is shown that if there are n primes, then there are $n + 1$ primes and so on and since there is a first prime, the number of primes must be infinite. It was

Maurolycus in his *Arithmetica* of 1575 who first explicitly recognised the method of proof by mathematical induction and used it to prove, for example, that $1 + 3 + 5 + \dots + (2n + 1) = n^2$. This method was acknowledged by Pascal in one of his letters and he himself used it in his "Trite du triangle arithmetique" in 1665 where he presents what is now known as Pascal's triangle. Eves (1983: 245) states how "... In Pascal's treatise on the triangle appears one of the earliest acceptable statements of the method of mathematical induction."

PMI stands for the Principle of Mathematical Induction and is Peano's fifth postulate for the foundation of natural numbers. According to PMI, a subset of the set \mathbb{N} natural numbers which contains 1 and the successor of n whenever it contains n must be equal to \mathbb{N} . The principle of mathematical induction may be stated and illustrated in the following manner: Let P_1, P_2, P_3, \dots be a sequence of statements with the following two properties.

1. P_1 is true.
2. The truth of P_k implies the truth of P_{k+1} ($P_k \Rightarrow P_{k+1}$).

Then the statement P_n is true for every positive integer n .

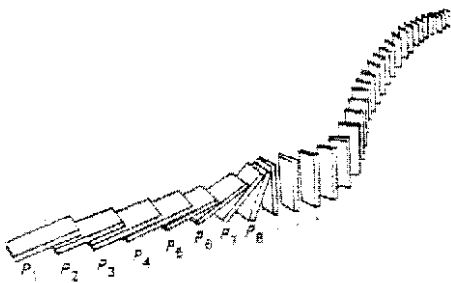


figure 8. (Fleming & Waring 1989 : 427).

The proof involves the application of the basic rule of inference known as Modus-Ponens. The previous illustration shows an endless row of equally spaced dominoes where the spaces between the dominoes is less than the length of the dominoes. If the first one is pushed down, it will knock down the second, the second one will knock down the third, etcetera and so as a result of a chain reaction the k th domino knocks down the $(k + 1)$ th one. Here two vital points are that the first domino falls and that if any domino falls then so will the subsequent one. Although Movshovitz-Hadar (1993) criticises this approach because it would not physically be possible to line up an infinite number of dominoes, this does seem to be an effective metaphor to use in this situation. Another illustration of the idea would be the

ladder analogy whereby if it is possible to climb onto the bottom rung of a ladder and, given any rung of the ladder it is possible to climb onto the next one, then since the rungs of the ladder are equally spaced, it should be possible to keep on climbing the ladder indefinitely, assuming there are infinitely many rungs. In the lessons in the subsequent sections, various visual representations will be used to introduce the concept of mathematical induction.

Attention has been given to different aspects of the learning and teaching of mathematical induction by many scholars. There are those who have questioned the exploratory potential of proofs by mathematical induction, the level of rigour reached and the contribution made to the deductive abilities of students. For example, Dubinsky (1986: 305) remarks "... it seems to be generally accepted by college mathematics teachers that, in general, our students do not come to understand this concept" and furthermore: "... very few students can actually construct an induction proof and not many can understand one."

In Mathematical Induction very often students write down the statement P_{k+1} , assuming that it is true, and work their way back to P_k . Since the steps in the procedure are reversible, they appear to have proved the result without understanding what they are doing. Sometimes students try to prove results meant to be proved by mathematical induction by simply stating that they are true for $n = 1$, $n = 2$ and so on. They need to recognise that an argument needs to convince not only themselves but the whole community and there are recognised ways of persuading others.

Understanding of proof by mathematical induction could be encouraged initially by providing numerous examples of apparent number patterns and generalisations. Students need to learn to understand an induction proof that appears in some larger discussion. Dubinsky (1986) observes that the ultimate goal is to make proof by induction part of the student's mathematical repertoire so that he/she has some idea of knowing when it can be applied without having to be told that this is an induction problem.

2.10.4 Conclusion

There are many wonderful patterns to be found in sequences of numbers but these can only be considered to be generally true if they can be proved. In this way pupils would be provided with the opportunity of rising to Van Hiele's theoretical learning level. Since proof by Mathematical Induction would give students insight into mathematical proofs and help them with possible future studies in this field, it would be beneficial to deal with it here. They would become aware of the fact that their conjectures need to be substantiated by acceptable methods of proof. The NCTM's Curriculum and Evaluation Standards (1989) expressed the need for more attention to be paid to proof by mathematical induction because although this is a very important tool to mathematicians, it has always been a difficult topic for students to master. In Chapter 4 the students will be required to find apparent rules from visual examples and then later on prove the correctness of their conjectures by means of mathematical induction.

2.11 Summation

Sequences and Series have evidently always been a relevant and fascinating mathematical topic. They involve patterns and throughout history mathematicians have discovered patterns, made generalisations and attempted to prove them. Orton (1999: 193) raises the question "... Do pupils have the capability to create a proof or justification for generalisation from number patterns?" This will form part of the investigation in subsequent chapters. Visual representations and patterns will be used to promote the advancement of student's thinking to subsequent thought levels in the topics of sequences and series. They will be given the opportunity to observe patterns, form generalisations and then prove them in the same way that mathematics has been developed throughout the ages. Phillip Jones (NCTM 1989b:11) points out how "The process of making successive extensions and generalisations has gone on over the centuries" and could now be viewed as a "pedagogical necessity."

CHAPTER 3

The learning theories of Piaget, van Hiele and Freudenthal and the importance of visualisation, exploring patterns and generalising in mathematics

3.1 Introduction

Before 1960 Zevenbergen (1993: 6) claims that there was "a strong emphasis on rote learning" but Piaget's theory had a strong influence on the changes which have occurred in mathematics education since then. As a result of his influence, teachers were encouraged to change their environments to enable children to engage in concrete experiences. This approach has been found to be effective not only for young children but, as pointed out by Schwebel and Raph (1974) the experienced teacher discovers the beneficial effect of the action-orientated approach applied to all age levels, ranging right from childhood to adulthood.

Van Hiele's theory originated from Piaget's findings and Pegg (1998) observes how both "... are relevant to, and designed to facilitate, school-learning activities, albeit in different ways." He continues to point out that "The van Hiele levels are a series of signposts of cognitive growth reached through a teaching / learning process as opposed to some biological maturation" (Pegg 1998: 337). The van Hieles suggest that at first students should be provided with activities involving new concepts and in an informal way make use of the insights or skills they already possess. Treffers remarks that "The aim is to acquire a rich collection of intuitive notions in which the essential aspects of concepts and structures are pre-formed. This, then is laying the basis for concept formation." (Murray et al 1988 : 177). Thereafter, students can gradually be introduced to mathematical terminology and more rigorous modes of reasoning as they become ready for it.

In this century the German mathematician Felix Klein (1924) made strong objections to the practice of presenting mathematics topics as completed axiomatic - deductive systems and was in favour of the bio-genetic principle in mathematics teaching. Michael De Villiers

remarked that this approach has also been supported by numerous others besides and added:

A genetic or reconstructive approach is therefore characterised by not presenting content as a finished (prefabricated) product, but rather to focus on the genuine mathematical processes by which the content can be developed or reconstructed (de Villiers 1998: 248).

Freudenthal strongly believed that learners should become involved in establishing theory for themselves and claimed that definitions, instead of being preconceived, are "... the finishing touch of the organising activity and the child should not be denied this privilege" (Freudenthal 1973: 417 - 418). His idea was that the concept and its properties should have been known for a while and only defined afterwards in such a way that the definition "... models new objects out of familiar ones" (Freudenthal 1973: 458).

Spatial and visual skills form an integral part of many aspects of mathematics. W.W. Sawyer (1964) advocated the importance of vision in school mathematics and suggested methods of visualising mathematics in order to gain the attention and comprehension of students. It seems that visualisation could form a vital part of the introductions to the topics of sequences and series, facilitate movement from one level to another and enhance thought processes while working at the various levels. In the lessons that follow in Chapter 4, numerous visual examples will be provided and learners will be encouraged to form mental pictures of the topics they study.

Moodley (1992) described the formalistic approach to mathematics education as one involving acquiring a body of knowledge whereas the processes approach consists of processes such as exploring patterns and generalising which will be studied here, Sawyer (1955: 12) refers to mathematics as "the classification and study of all possible patterns" and Williams and Shuard describe searching for order, and pattern as being "one of the driving forces of all mathematical work with young children" (Orton 1999: vii). Commenting on the results of some research done by Fischbein, Roper remarks that "These conclusions suggest very strongly that the urge to seek for order, for pattern, for a deterministic solution is a

compelling one, and one which grows with age" (Orton 1999: 171). This seems to suggest that children of all ages be provided with a much wider array of pattern structures and tasks within the classroom. The provision of numerous patterns could encourage students to become accustomed to searching for patterns, extract more meaning and enjoyment from their learning environment, employ a variety of strategies, approach new tasks on their own and engage in challenging problem solving.

The establishment of patterns could lead to the desirable goal of generalising regularities found. W. Dörfler in his generalisation model in Streefland (1991) regards the symbol as being removed from the original context and becoming the object of thought. It could then be checked to determine whether results hold in other situations. In the lessons of Chapter 4 learners will initially be encouraged to obtain an expression of generalisation by focusing on visual similarities between figures. Proof by mathematical induction will also be included as a high level activity to encourage them to verify their findings in a mathematically acceptable manner.

The topic of levels of learning, particularly those of van Hiele and Land as well as the influence of visualisation, exploring patterns and generalisation will be investigated in this chapter. It has been said that "... the way mathematics is taught is more important than the topics covered" (Phillippou 1998: 2). The research here should help to promote "... following the process of creation of mathematics and to create a climate of search and investigation" (Phillippou 1998: 2) where the search is aimed towards reaching a specific goal.

3.2 Piaget

3.2.1 Introduction

The famous Genevan Professor Jean Piaget (1886 - 1980) was born in Switzerland and devoted his life to studying human intellectual development. He became involved in Freud's techniques and Alfred Binet's intelligence tests which led to an interest in the mistakes being

made by children. There are two main periods associated with his work. The middle period involves the development of stage theories. Later in life he returned to his earlier work concerning how learning occurs.

3.2.2 Piaget's stages of intellectual development

Zevenbergen (1993) refers to the four stages of development of Piaget :

Sensorimotor stage (0 to 2 years)

This is the stage which involves practical knowledge in which senses and perceptions are co-ordinated with movement and action and there is little anticipation of consequences of actions.

Pre-operational stage (2 to 7 years)

At this stage the child is egocentric, concentrates on certain functions only, is unable to reverse mental actions, relies on intuition rather than reasoning, has connected but illogical and inconsistent ideas and engages in free and symbolic play.

Concrete operational Period (7 to 12 years)

Here the perspectives of others are considered; conservation of number, seriation, length, area, volume and mass appear; more than one aspect of problems can be appreciated; mental actions can be reversed; the perceptual and physical help in the thinking about problems; games involving rules materialise.

Formal operational Intelligence (12 years and more)

This stage coincides with adolescence when the child is able to work with abstract ideas, test deductive hypothesis, use verbal as opposed to concrete propositions in problem solving such as $a > b$ and $b > c$ implies $a > c$ and is able to appreciate metaphors.

3.2.3 Characteristics of Piaget's theory

Piaget stresses that these stages are ordered in time but not that the ages as specified are fixed. He does not deny that differences in ability do exist but he is chiefly concerned with the intellectual development of all children in general. The stage which a child has reached is determined by the most advanced activity which he is capable of performing. Once a child is

in a fairly advanced stage (e.g. the formal operational stage) he should still be able to perform all operations attributed to the previous stages. Experiments he conducted have shown that it is very difficult, if at all possible, to train a child to advance from one stage to another when he or she is not ready for it. In addition, it seems that knowledge acquired as a result of special training cannot be the same as knowledge acquired naturally by means of experiences in which the individual himself becomes actively involved.

The operational structures between the ages of seven and eleven are called concrete because their starting point is always some real system of objects and relations that the child perceives i.e. the operations are carried out on concrete objects. During the formal operational stage of development the child is further released from the world of physical objects. He is now capable of hypotheses or ideas. As he constructs new operations, he attains new mental structures and is able to consider all possibilities or combinations rather than those based on experience or experiment. His thoughts, now able to follow hypothetical-deductive procedures, are no longer tied to existing reality.

3.2.4 Mathematics associated with different stages of Piaget

Different aspects of Piaget's theory are related to the various mathematical concepts associated with particular levels of schooling. At the preoperational level or pre-school level plenty of free play and concrete materials play an important role in the development of mathematical concepts. During the concrete operational level stage, concrete materials begin to give way to numeric symbols. The child begins to be able to apply logical and mathematical thinking to concrete situations. At the formal operational level, when abstract thought becomes possible, numeric symbols may be replaced by algebraic ones as algebra and logic become understandable.

The subject of algebra is first introduced to children at high school. Patterning and generalisation is of major importance in this regard and Zevenbergen (1993: 3) comments how: "... Algebra requires that students need to be able to see patterns and express

generalisations before they can be successful with algebraic symbolism."

The age of students at high school coincides with the formal operational stage of thought when students are supposed to be capable of forming generalisations by means of using symbols instead of numbers. However, certain students have great difficulty in appreciating algebra even at this stage and so in this study attempts will be made to consider closely the stages involved in developing algebraic concepts and how to facilitate movement from one level to another. Students need to understand their algebra and not merely memorise it or operate with symbols as meaningless objects.

3.2.5 Types of knowledge

Piaget refers to the growth of logical knowledge as development and the growth of physical and social knowledge as learning. His theory of knowledge recognises physical knowledge, logical knowledge as well as social knowledge and representation. He does not believe that logic arises out of language but rather that it is caused by actions. Once actions have been repeated several times, they become generalised and form assimilation schemes, which are very similar to the laws of logic. For this reason, pupils, especially at a young age, should experiment with objects in order to form arithmetic, algebraic and geometric concepts. Piaget (1973) recognises two different types of experiences resulting from activity called physical experience and logico-mathematical experience. The former involves such activities as the weighing and measuring of objects while the latter (of which few people are aware) involves not the physical properties of objects but the co-ordination of actions carried out upon them. Unlike Aristotelian abstraction which is derived from the physical properties of objects, logico-mathematical abstraction could be called reflective abstraction as mentioned by van Hiele (1959), because it reflects what was on a lower plane to a higher plane of thought and reconstructs at a higher level everything that was drawn from the co-ordination of actions.

3.2.6 Implications for mathematics teaching

Piaget believes that there are logico-mathematical structures which develop gradually and naturally in children's minds. His work suggests that pupils should become actively involved in the learning process and that teachers should be aware of the stage of development of each of their pupils. He favours group work as pupils are able to learn by teaching one another. This approach will be promoted in Chapter 4 in which students will be divided into groups and be encouraged to become actively involved in the lessons, engage in discussions and discover results for themselves.

3.2.7 Conclusion

Piaget's Theory of Intellectual Development suggests that any new concept be taught by progressing through the various stages of intellectual development (even to pupils who are already in the formal operational stage). The concept of a sequence is met formally for the first time by pupils in Grade 12 although they would have come across it at various stages of their schooling. Hence, in this case, a range of experiences, such as introductory activities involving tossing coins and folding paper as well as pictorial illustrations could be arranged. Later numerical examples are encouraged, formulae derived and eventually results proved and utilised in various contexts.

3.3 Van Hiele

3.3.1 Introduction

After Piaget, more consideration continued to be given to levels of learning in mathematics. Pierre van Hiele and Dina van Hiele-Geldof devised a theory to address the problem of learning geometry. In their theory presented in 1959, they analysed both the teaching and learning of geometry as well as the interaction between the two procedures. This evolved because of their concern that at the time high school teachers in the Netherlands were finding that their pupils were not performing very well in geometry. Land (1990) notes that the three components of the van Hiele's theory include: insight, the levels of thought and phases of learning.

3.3.2 The van Hiele model of levels of geometric thought

The Van Hieles believe that students progress through different levels of thought in geometry. Beginning with a Gestalt-like visual approach and then passing through levels of increasing sophistication including description, analysis, abstraction and proof. During his later research, the visual, descriptive and theoretical levels come to be used to describe the discontinuity of the learning process. However, here the five levels of geometric thought of the original van Hiele model, distinguished by certain characteristics described by Holmes (1995: 332) are listed below. These levels are named levels 0 to 4 although some texts label them as levels 1 to 5.

1st level - level 0 - visualisation

Initially children recognise figures (for example a square) by their appearance but they may not be able to recognise their properties (such as right angles). They can identify many figures and perceive a figure by its shape as a whole. But irrelevant attributes, including the orientation of a figure, can cause them not to recognise a shape anymore. At this level children are able to represent figures as visual images and they see objects as classes of figures (for example, triangles). They do not pay attention to geometric properties or characteristic traits of classes of figures but name, recognise, compare and operate with geometric figures according to their appearance.

2nd level - level 1 - analysis

At this level children are able to recognise and characterise shapes by their properties and they begin to analyse geometric figures. They no longer see figures merely as visual Gestalts, but as possessing a collection of properties. Not only do they become aware of attributes of figures, but they develop concepts of figures as a result of internationalization of the properties they have noticed. They can, for example, say that a triangle has three sides or a parallelogram is a four-sided figure with both pairs of opposite sides and angles equal. But since they have not constructed relationships between figures, they would not for instance know that a rectangle is a special type of parallelogram. They often include redundant information in their definitions, stating, for example, that a rectangle has four right

angles and two pairs of opposite sides that are both equal and parallel. Although children can memorise definitions at this stage, they would rather discard them and use their own meaningful descriptions.

3rd level - level 2 - informal deduction

At this level precise definitions are understood and accepted by students. They are able to refer to definitions when they study figures and are able to understand the relationship between the properties of different figures. They are able to reason, for example, that because a square has all the properties of a rectangle, it is a rectangle too. They appreciate the concept of class inclusion, recognising that a square is a rectangle, which in turn is a parallelogram which is a quadrilateral. Although they are capable of "if then" thinking, they are not as yet able to construct proofs. However, they are able to distinguish between necessary and sufficient sets of conditions for a concept and can grasp and sometimes even provide logical geometric arguments. As a result of reasoning about properties of classes of figures and interrelating properties of figures and classes of figures, they are able to reorganise their ideas and further their understanding in geometry.

4th level - level 3 - formal deduction

When students reach this level, they can establish theorems within an axiomatic system and rely on proofs to determine the truth of mathematical statements. They are able to distinguish between undefined terms, definitions, axioms and theorems and are capable of constructing original proofs. The type of reasoning attained at this level results in the establishment of second-order relationships. This level of thought is necessary for success in high school geometry courses.

5th level - level 4 - rigour

It is at this level that students reason formally about mathematical systems. They can study geometry without reference models and are capable of reasoning by formally manipulating geometric statements such as definitions, axioms and theorems. They have reached the level of thought of mathematicians as their reasoning leads to the establishment, elaboration and comparison of axiomatic systems of geometry. This is an extremely abstract level of thought in geometry which many people may find unattainable.

3.3.3 Characteristics of the van Hiele model

According to the Van Hiele theory, learning is a discontinuous process with apparent jumps in the learning curve. This indicates the existence of qualitatively different levels of thinking. Since the van Hiele thought levels are sequential and hierarchical, learners have to move through them in order. Once a student has developed understandings at one level, it becomes possible for him or her to proceed to the next level. At any one particular level the concepts developed are integrated and extend to develop thought at the subsequent level. According to Mayberry (1983) a student may operate at different levels depending on the concept in question. It seems that advancement from one level to the next depends on instruction rather than age or biological maturation. Providing pupils with appropriate activities enables them to advance to higher levels.

The van Hiele theory indicates that there is a characteristic language for each level. This means that people reasoning at different levels can neither understand nor follow the thought processes of the other. In order to reach higher levels of geometric thought, many opportunities of engaging in activities that reveal the content and relations of a particular level are required. Furthermore, it is necessary to discuss ideas represented in the activities, solve problems and integrate ideas through review and summarisation. Learning does not occur if the level of instruction does not match a student's level of development. It is of great importance that teachers do not make use of content, materials and vocabulary of a level which is too high for their pupils to understand. The resulting anxiety and frustration of students could impede their further advancement.

3.3.4 Structure and insight

Van Hiele commented that he was convinced that the purpose of teaching mathematics was not merely to enable students to learn facts but rather for the "... development of insight" (van Hiele 1986: 4,5). He believes that "Insight exists when a person acts in a new situation adequately and with intention" (van Hiele 1986: 24). Sawyer regards insight as a "perception of structure" Land (1990: 22).

Each of the levels, with the exception of the base level, has what might be termed a "structure" or "network of relations" Land (1990: 22). Insight arises when a student perceives this structure and is able to continue, explore and expand it as well as see it as part of a greater structure.

3.3.5 The application of the van Hiele learning and teaching theory to algebra

3.3.5.1 Introduction

The Van Hiele approach, which has been successfully applied to the instruction of geometry in the Netherlands and the Soviet Union, seems to warrant some application in algebra too. The application of the van Hiele theory to the learning of algebra and the learning of sequences and series in particular will be considered in this and subsequent sections. Land (1990: 25) in speaking of the original five levels, quotes Usiskin as follows: "Furthermore, it has been found that the fifth level is either nonexistent or untestable" because it is generally too high for secondary school students. However, in subsequent examples, several writers incorporate all five levels in their descriptions of the development of mathematical thinking.

3.3.5.2 Van Hiele levels in history

Freudenthal notes the relevance of the van Hiele levels both in the order in which mathematics is learned by students and the historical development of mathematics :

History moved according to these (van Hiele) levels. The complete induction was exercised since antiquity; the 'side-and-diagonal' numbers are a profound application of this principle. The first man who grasped the principle consciously and formulated it was Pascal. The formulation, a noteworthy feat, required quite new linguistic means (Freudenthal 1973: 123).

Various topics in the above quotation have been discussed in the second chapter and will be covered in the lessons contained in Chapter 4 and the appendix. The van Hiele levels and teaching phases will be taken into consideration in the development of these lessons.

The relevance of the van Hiele levels in algebra, can be seen in the following description given by Bell of the emergence of the concept of an abstract group:

The entire development required about a century. Its progress is typical of the evolution of any major mathematical discipline of the recent period; first the discovery of isolated phenomena [basic

level-visual]; then the recognition of certain features common to all [*second level-description*]; the search for further instances, their detailed calculation and classification [*third level-theoretical*]; the emergence of general principles making further calculations, unless needed for some definite application, superfluous [*fourth level-deduction*]; and the formulation of postulates crystallizing in abstract from the structure of the system investigated [*fifth level-rigor*] (Land 1990: 27).

Van Hiele (1986) believes that the various thinking levels are inherent in man and not only in geometric thought. Thus the above-mentioned pattern may be applied to the formulation of other mathematical concepts such as sequences and series.

3.3.5.3 Land's levels in the teaching of functions

In his correspondence with Land (1990: 131), van Hiele (April, 1989) indicated his belief that there is no visual level in algebra: "What you call visual is quite another thing than the visual level of geometry. What you call visual cannot be called like that in the psychological sense.

Algebra has no visual level" Van Hiele later wrote "Everything at the basic level can be learned by simply pointing to and giving the name". He continued to suggest to Land that

... Before your first column you add one in which the descriptive level has not been attained. You can try to answer the question: What problems can children be given before they have attained the descriptive level, which can be understood by pointing to and which may lead to graphs (Land 1990: 131-132).

In agreement with van Hiele's suggestion and the approach followed by Land, a pre-descriptive level called level 0 will be used here which is not a true van Hiele level. At this level students will be presented with numerous patterns and visual representations of sequences in order to enable them to recognise a sequence when they see one in various contexts. Land (1990) comments on how research indicates that the provision of visual experiences enables both the left logical and analytical side of the brain as well as the right spatial holistic hemisphere to be utilised.

3.3.6 Objects

A category is composed of a set of elements called objects and a set of relations between the objects called morphisms which satisfy certain postulates. Hoffer (1983) regards each van Hiele level as a category with different objects at each level.

The first level: level 0: objects : base elements of the study.

The second level: level 1: objects : properties which analyse base elements.

The third level: level 2: objects : statements that state the properties.

The fourth level: level 3: objects : partial ordering of statements that relate the properties.

The fifth level: level 4: objects : properties that analyse the partial ordering.

Hoffer (1983) proposed that if the van Hiele model is to be used for designing learning experiences in other structured subjects besides geometry then the objects for each level should be determined. These are the objects perceived by the student and suggested by the mathematical topic. The objects involved in sequences and series will be listed in Chapter 4.

3.3.7 Land's adaptation of van Hiele's levels in algebra

Land (1990) considered the application of van Hiele's theory to the teaching of functions. She consulted with van Hiele and decided to use the first 4 of the original 5 levels. As far as the fifth level (level 4) is concerned, Land (1990: 174) comments how van Hiele wrote "It seems [to] me unreasonable to investigate about the deductive level: Hardly anyone ever attains it." Thus the fifth level (level 4) will not be considered in the development of the mathematical thinking of matriculation students in sequences and series either.

1st level - level 0 - pre-descriptive level

In Land the base elements, exponential and logarithmic functions, are the objects of study. This is not a true van Hiele level. An example would be the graphing of the exponential function $y = 3^x$ by substitution (a skill brought to the topic by students).

2nd level - level 1 - descriptive level

The objects here are the properties of exponential and logarithmic functions which are established inductively. Elementary symbol manipulation and formulation of expressions as well as interpretation of parameters is possible at this stage. For example $f(x) = 2^x$
 $f(x) = (\frac{1}{2})^x$ or $f(x) = 2^{-x}$, $f(x) = 2^x - 1$ and $f(x) = 2^{x-1}$ can be drawn without plotting points (as students can appreciate the effect of changing the values of a, b and c in $f(x) = a^{x+b} + c$.)

3rd level - level 2 - theoretical / informal level

This level has the relationships between the properties of exponents, logarithms and exponential / logarithmic functions as its object. Words may be accurately and concisely defined and a relevant deductive argument can be followed. Derivation of the mathematical formula $A = (1 + \frac{i}{m})^{nm}$ is possible and the relationship between $f(x) = 10^x$ and $g(x) = \log x$ can be explained. Equations such as $\log_6(x - 5) + \log_6 x = 2$ can be solved too.

4th level - level 3 - theoretical / formal level

Objects here include the partial ordering of statements relating properties of exponential and logarithmic functions. The student understands the significance of deductions, postulates, theorems and proofs. Symbols are used with insight and understanding to construct a proof or solve a problem. Verbal summaries of an argument may be made.

Examples include :

Prove : $\log(x - \sqrt{x^2 - 1}) = -\log(x + \sqrt{x^2 - 1})$

Solve for x : $\log_x 1 < -1$.

Summary lists of objects and characteristics of the above four levels associated with sequences and series follows in Chapter 4.

3.3.8 Instructional phases

Dienes believes that learning mathematics is not really a stimulus-response learning situation because "... the accent in mathematics is more on structure and less on content" Dienes (1968: 19). He recognised six instructional phases including freeplay, games, search for commonality, representation, symbolisation and formalisation.

Van Hiele too has identified his own instructional phases. These include: information where the student becomes acquainted with the content; guided orientation where students familiarise themselves with ideas; explication when students became aware of intuitive relations and start elaborating on their intuitive knowledge; true orientation when students are capable of choosing their own activities and orientating themselves within the framework of

relations; integration when students are able to summarise, integrate, reflect on, describe and apply what they have learnt.

In their questionnaires in the lessons to follow in Chapter 4 the students will be asked to summarise and reflect on concepts they have learnt and draw comparisons between different types of sequences they have encountered. More challenging questions, often involving the integration of different concepts learnt, will also be provided. Land believes that algebra has a large verbal and visual component. Students need to form and understand concepts before "... they can realize the need for precision in the expression of these concepts" (Land 1990: 109). Van Hiele has a "telescoped teaching" method (Land 1990: 175) which means passing through the phases of learning again when a child does not understand something.

3.3.9 Implications of van Hiele's views for secondary school in mathematics teaching

Teachers need to become aware of the levels of their pupils so as to provide suitable instruction for them. Interactive teaching approaches are most suitable because they not only help pupils construct concepts as a result of their own actions and reflections, but they also help them to refine their ideas as they communicate about their findings. Assessment activities help teachers to be aware of the level of thought of their pupils and to plan future instruction in order to help them move to higher levels of thinking.

Van Hiele believes that in order to move from level to another, it is important that a learner becomes fully acquainted with objects belonging to the former level. His model seems to have relevant applications not only for the teaching and learning of geometry but the teaching and learning of algebra as well. Van Hiele (1986: 52) himself believes that his model is suitable for the learning of algebra. He regards his theory as a theory of levels of thinking and believes that in every topic the student's conceptualisations about the topic go through different discontinuous levels. This happens in such a way that the characteristics of the concepts and the type of activities that can be performed at each level are unique to that

level. The levels are hierarchical and meaningful activity on a particular level is required to move to the subsequent one.

Van Hiele considers the initial level of thinking about a new topic to be a stage when the student is exposed to the topic in such a way that as a result of his own personal experience and observation, he can construct his initial concepts. This may involve using concrete apparatus but only if the concrete material can provide true experiences of the relevant concept. At this stage the teacher should encourage discussion but the language should be informal since the standard terminology is introduced gradually after the student has constructed the concepts. Once the student is able to operate formally with the concepts as well as symbolise and analyse then, he is said to have reached the next level of thinking. The intuitive foundation of proof begins with a pupil's belief that the truth of some assertion is connected with the truth of other assertions. Van Hiele (1986: 124) explains how by analysing and exploring these laws, logic is created and the rational foundation for proof is formed. Proof by mathematical induction will be considered in the last of the subsequent lessons.

3.3.10 Van Hiele and Piaget

Piaget regards biological maturation as important in the learning process while the van Hieles place more emphasis on the type of instruction employed. Van Hiele is opposed to accepting that logico-mathematical experience results directly from actions but that the first "abstraction" (van Hiele 1959: 23) or transition to a higher level results from actions or relations on objects regarded as what he terms "... thinkable unities" (van Hiele 1959: 23). He criticises Piaget's statement that the arithmetical thinking of a child goes hand in hand with his logical thinking as this would imply that a child would have reached the stage of logical thinking some time before reaching the age of secondary education. He notes that there are various stages in the development of logical reasoning, where at first it is the visual appearance which reflect these properties but later the name of the figure itself will adequately reflect them. The final stage is reached when a structure is seen as

"... a planned and interrelated collection of properties" (van Hiele 1959: 8).

Piaget made a great contribution to the sensori-motor area of learning. According to stereometry, there are certain concepts that only come about after a period of both touching and doing. Although it is true that sensori-motor development is most commonly encountered by very young children, the stages mentioned by Piaget are not necessarily connected to a particular age but are typical for numerous learning processes, regardless of the age at which it takes place. Van Hiele (1959 : 14) writes:

The learning process in stereometry in the case of pupils of about 16 years can to a very large extent be aided by allowing the pupils to use tangible material. The abstract structures of stereometry rest for a large part upon experiences with concrete material, but it is not absolutely necessary that these experiences be undergone at a very young age.

3.3.11 Van Hiele's theory and the SOLO taxonomy

Van Hiele's theory has been useful in identifying problems students experience in geometry as well as evaluating the structure and development of secondary school texts and syllabi. However, problems can arise because of the restricted nature of the level descriptions. For example, it might be difficult to determine the levels of notions not directly connected to properties of figures, class inclusion and deduction. Pegg (1985) refers to the criticism about the discontinuous nature of the levels, their simplistic one-dimensional nature and the way in which they do not explain the diversities in the behaviour of students.

In addition to the van Hiele theory, there is the SOLO Taxonomy. The difference between the theories is described by Pegg (1998: 337) as follows:

The van Hiele levels are a series of signposts of cognitive growth reached through a teaching / learning process as opposed to some biological maturation. SOLO, however, is particularly applicable to judging the quality of instructional dependant tasks.

SOLO represents a shift from Piaget and van Hiele's ideas as it describes responses, not people and judges the quality of instructional dependent tasks. It measures the attainment of a student at a particular time and place and the main difference between SOLO and van

Hiele's theory lies in the conclusions drawn regarding the overall nature of the student's level of thought. Pegg (1998) has used SOLO to attempt to broaden the level description of van Hiele's original theory. Nevertheless, he notes that "... Despite controversy, the van Hiele levels continue to be used as an important framework for interpreting students' understanding of geometrical ideas" (Pegg 1998 (3): 335).

3.3.12 Conclusion

Van Hiele's theory seems to provide an approach to teaching any topic in mathematics. Van Hiele (1959) claims that bad results in the teaching of geometry is almost always due to the neglect of the teaching levels. In algebra he believes that the teacher aims at algorithms too soon and so the students attempt to learn them without being aware of any special functions they might possess. Land (1990) quotes Han's belief that the van Hiele model can be used in most structured disciplines. The teacher is required to interpret any topic in terms of the model.

3.4 Freudenthal

3.4.1 Introduction

Hans Freudenthal (1905 - 1990) was a mathematician who believed in mathematics for all. Teaching his two young sons arithmetic made him reflect on arithmetic at a primary level and caused him to write his didactics of arithmetic in 1942. During the nineteen-sixties Freudenthal became a member of the Modernisation Committee of the Mathematics Curriculum. His name has been closely linked to the Wiskobas project which began in 1968. This led to the development of realistic education at primary schools in the Netherlands where it is the main approach followed in mathematics teaching. He was the founder of the so-called realistic mathematics education.

Freudenthal has expressed the view that mathematics involves growing, developing, looking for problems, solving problems and organising subject matter. However, once these results are listed, the original character seems to be lost. According to Freudenthal (1973: 413) :

A great part of mathematical activity today is organising. We like to offer the results of our

mathematical activity in a well organised form where no traces betray the activity by which they were created. This organisation is a habit of mathematicians from oldest times. It is a good habit and a bad one. We freeze up the result of our activity into a rigid system, because this is our objective, because it is rational, and because it is beautiful, and this we teach.

Freudenthal was very concerned that the sources of insight be kept open for the learner in the teaching-learning process so that students be kept aware of the origin and development of the mathematics they are studying. He also stressed the importance of intertwining or interweaving learning strands. He was impressed by the pedagogue Comenius whose didactics in modern terms he believed would be "... The best way to learn an activity is to perform it". (Freudenthal 1973: 110). Thus Hans Freudenthal believed that the emphasis should be shifted from teaching to learning and from the teacher's activity to the pupil's.

3.4.2 Freudenthal and Piaget

Although Freudenthal really valued the observations Piaget had made about children, he was strongly opposed to his later more theoretically orientated work. He felt that Piaget had often been wrong in his experiments and theoretical analysis. He was aware that Piaget was not a didactician but instead was preoccupied by knowledge and the structure of the human mind. He believed that Piaget had not taken the influence of language into consideration in his clinical interviews with individual children. Nevertheless it is true that Piaget gave Freudenthal not only "... food for thought but also inspired him to do tests" (Streefland 1991: 36).

3.4.3 Freudenthal and the van Hiele

Van Hiele was once a student of Freudenthal and had a Gestaltist point of view regarding the problem of what to attempt to teach in mathematics. Freudenthal was influenced by the van Hieles and their work regarding learning levels and their importance. He believed that "... the learning of a mathematical concept or skill is a process which is often stretched out over a long term and which moves at various levels of abstraction" Streefland (1991: 24). Freudenthal felt that the van Hieles deserved all the merit for the discovery of levels but that his own idea of levels differed from theirs as he regarded his levels as being "... relative rather than absolute" Streefland (1991: 37).

Thus Freudenthal believed that the learning process is structured by levels and that it is the activity of the lower level that becomes the object of analysis of the higher level. The pupil learns to mathematise his spontaneous activities. Although algorithms are of major importance in mathematics, their danger lies in teachers being tempted to teach them without allowing pupils to reinvent them. Students may even be as captivated by a well-functioning algorithm as by a game. It is possible to teach a skilful pupil above his own level by means of algorithms without any reference to the sense of the actions and to provide application patterns too. However, this knowledge probably will not be retained for a long term and it would probably be easier and definitely more beneficial to raise the pupil to the required level than go through a clumsy didactic process.

Freudenthal also recognises the discontinuity or jumps in the learning curve mentioned by the van Hiele's. Goffree comments how the van Hiele's noted that on a lower level things may be described as being done "... intuitively, informally and acting with objects, but this is regarded differently at a higher level" (Streefland 1993: 27). Freudenthal, however, attributes the rise in levels to what he terms reflection. He believes that continued learning requires reflection and that if mathematics is taught as a reinvention, then the child is prepared to move to the next level of thought. Freudenthal (1973: 130) remarks:

Reinvention that is a didactic principle on research level, should be the principle of all mathematical education not only on the bottom level where it is too near to manual playing to show pronounced mathematical features.

His idea of teaching mathematics as a reinvention will be employed in lessons in Chapter 4.

3.4.4 The teaching of algebra

A.W. Bell (1983: 283) states "... the way in which mathematics exploits the spatial properties of its symbolisms and develops 'manipulations' of symbolic expressions is a spatial characteristic which it does not share with ordinary language." Freudenthal holds that all language instruction requires a vast and well-directed reading drill but that algebraic expressions are a linguistic matter with a peculiar structure which can be a great deal more

complex than ordinary linguistic matter. Algebra should be taught and algebraic texts should be built up according to a well-designed plan based on increasing degrees of difficulty with not too much stress laid on substitution. Freudenthal points out the danger that, in the teaching of algebraic formulae, the algebraic matter could become a meaningless game. The provision of realistic problems and problems that test not only whether the student can apply some formula or rule but also whether he does so consciously or knows why he is doing so helps to ensure that algebraic learning becomes more meaningful.

Freudenthal's approach is not to teach students a piece of formalised mathematics but rather to accompany students in a learning process in such a way as to make them conscious of it. The content, for example, the mathematics of Pascal's triangle, is not merely the topic but the understanding of the meaning of formalising and algorithmising in mathematics. The natural development of mathematical language in phases of abstraction or formalisation can be demonstrated so that the student goes through various learning processes to ascend to the levels of the language required rather than being immediately served with the finalised linguistic form. The type of approach in Chapter 4 will be to encourage students to build up the necessary language as they are introduced to new concepts and establish rules for themselves.

3.4.5 Sequences

Freudenthal states that

The number sequence is the foundation-stone of mathematics, historically, genetically, and systematically.

Without the number sequence there is no mathematics. (Freudenthal 1973: 171, 172).

He adds that:

Children first learn to count, then later they learn to count in twos, threes, triangular numbers, squares, so it goes on - up to the power series, well ordering types, and recursive functions (Freudenthal 1973: 173).

Furthermore, he describes how the positional notation provides a simple model for the abstract mechanism of the number sequence which is so simple that it can be taught to both children and machines. The child learns to appreciate the notion of infinity, described by

Freudenthal (1973: 173) as "... the alpha and omega of mathematics". Limits and infinite series all form part of the topics dealt with here and Freudenthal (1973: 174) remarks:

"... Do not forget that in this stage all infinite processes, from the generation of the number sequence to the convergence of infinite function series, are imagined as reeling off in time."

Freudenthal (1973) points out how Pascal's triangle, described previously, shows an obvious

pattern such as the one leading to the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for the binomial coefficient.

He comments that everything done with natural numbers, if it is to hold for them, needs complete induction and points out that "Complete induction applies to the construction of a sequence $\{a_n\}$ if by some recipe to every element of the sequence the next is given" (Freudenthal 1973: 183). However, he believes that many school mathematics educators aim too high for their pupils and need to take their students' levels of thought into consideration in their teaching of topics such as sequences and series.

3.4.6 Conclusion

Freudenthal himself made a great contribution to mathematics in the fields of Homotopy Theory and Lie Groups in Group Theory. Furthermore, he contributed a wealth of ideas and conceptual tools to the development of mathematics education. His ideas, too, will be taken into consideration in the compilation of lessons on sequences and series in subsequent chapters. Alan Bishop, in his preface in Freudenthal (1991), comments that he

... shines through all his writing as a beacon to all of us in mathematics education, and in education generally. His writing focuses as always on the essential concerns of mathematics education, as he saw them, and children are our principle concern. (Freudenthal 1991: ix).

3.5 Visualisation

3.5.1 Introduction

Visual thinking has become very prominent in modern day life. Tables, formulas of numbers and symbols are giving way to dynamic visual presentations by means of computers. Visual thinking is required to understand, analyse and predict. Hershkowitz and Markovits

(1991: 38) claim: "... It seems that visual thinking will be the primary way of thinking in the future". Since visual comprehension plays an important part in much of what we do, visual skills need to be reinforced at school. In mathematics these are certainly relevant because they contribute to the establishment of mathematical concepts.

3.5.2 The use of visualisation in mathematics

Visualisation may be described as seeing with the mind's eye or having a mental picture.

Dwyer notes that: "... The use of visuals specifically designed to complement printed instruction can significantly improve student achievement of certain types of educational objectives" (Dwyer 1988: 365). But the student's prior knowledge level in the content area, the relevant level of processing of information and the academic capacity of the students in question will all influence the type of visuals which will be most efficient.

However, Dwyer (1988: 365) observes that:

Although properly designed and positioned visuals can significantly improve student achievement of specific types of educational objectives, visualization itself represents only a mild rehearsal strategy which will not always optimize student achievement of the more complex levels of learning.

Nevertheless it does seem that visualisation may serve a valuable purpose in setting a solid foundation both in pre-instructional strategies and the "retrieval phase of the teaching-learning process" Dwyer (1988: 365).

3.5.3 The relevance of visualisation in the theories of Piaget, van Hiele and Freudenthal

3.5.3.1 Piaget

Piaget advocates a concrete approach. However, his theory does not support merely showing children audio-visual representations of objects but stresses that the child becomes actively involved in the learning situation. Learners need to have concrete material experiences and in the words of Piaget be able to "... form their own hypotheses and verify them or not verify them themselves through their own active manipulations" (Schwebel and Raph 1974: x).

3.5.3.2 Van Hiele

Visual skills can also be extended to mean the "... ability to interpret figural information", "... to read, understand and interpret the special symbols and conventions" and to manipulate "... objects within one's mind" Davey & Holliday (1992:27). Hoffer (1981) associated five skills with each of van Hiele's five levels. These include "visual, verbal, drawing, logical and application skills and since the five skills appear in some format at each van Hiele level, there are 25 cells which need development" (Davey & Holliday 1992: 26). This suggests that visualisation plays a significant role at each of van Hiele's levels and so it will also be included at each level when his theory is being applied to algebraic topics in Chapter 4.

The van Hieles suggest that at first students should be provided with activities involving new concepts and in an informal way make use of the insights or skills they already possess. Treffers remarks "The aim is to acquire a rich collection of intuitive notions in which the essential aspects of concepts and structures are pre-formed. This, then, is laying the basis for concept formation" (Murray et al 1998: 177). Thereafter learners could gradually be introduced to mathematical terminology and more rigorous modes of reasoning as they become ready for it. Now it is generally accepted that van Hiele's first level experiences need not be concrete or real-world problems but any tasks or experiences with which a group of students is familiar. Van Hiele supports the use of authentic rather than contrived situations to illustrate mathematical topics. Pegg (1985: 8) remarks that, according to van Hiele's theory, "While the visual aid does not lose its value as we proceed through the levels, the pupils' relationships to the figure changes".

3.5.3.3 Freudenthal

Freudenthal too holds that understanding can be promoted and deepened by visualisation. For example, he advocates that number lines are useful in teaching number concepts and favours the use of visual illustrations such as pictures, graphs and diagrams to promote the understanding of mathematical concepts.

Mathematising refers to "... turning a non-mathematical matter into mathematics, or a mathematically underdeveloped matter into more distinct mathematics"

(Streefland (1993: 72). The concept of mathematising is composed of two components:

"horizontal, that is, from the world in which the learner lives to what is still a world of symbols to him; and vertical. that is within the world of symbols" (Streefland 1993: 72).

Freudenthal (1991: 56) describes as an instructional principle to make use of "concrete, if possible, visual situations". Further, he mentions making use of the learner's current reality that is appropriate for horizontal mathematising (making a problem field accessible to mathematical treatment or mathematical in the narrow formal sense) and also offers both the means and tools for vertical mathematising (which effects the more sophisticated processing). This could be otherwise described as making use of different levels to teach the concept. For example, six times seven (things) may be mathematised horizontally by the rectangular scheme of 6 rows of 7 each. In vertical mathematisation, however, it is read as part of the sequence 7; 14; 21; 28; 35; 42;..... In the case of Pascal's triangle, looking at the triangle, numerous relations between its elements are obtained by horizontal mathematisation. But the visual algebraic expression of the binomial coefficients requires vertical mathematisation as do the combinational problems related to Pascal's triangle. The steps of induction required for proving relations are of a vertical character, although in the long run they are expressed horizontally.

Freudenthal's views were not only to "incorporate everyday reality emphatically in mathematics education, but especially also his fundamental idea to let the rich context of reality serve as a source for learning mathematics" Treffers (1993: 89). Freudenthal encourages the promotion of visualisation in mathematics teaching as he sees reality as being both a source and an application area in the learning of mathematical topics.

3.5.4 Visualisation in the instructional process

For many years Zimmerman (1986: 22) claims that American educators have emphasised the importance of the individual student in learning and achievement. They believe that learning is something that happens not to students but by students. Their assumptions is that in order for learning to occur, "... students must become proactively engaged at both a covert as well as an overt level" (Zimmerman 1986: 22). They are thus strongly in favour of self regulated learning. Their perspective has caused them to turn to real world contexts to achieve self-designated goals.

The fact that many learners do experience difficulty in learning from printed instruction suggests the use of visualisation to complement printed instruction. This has become a relevant instructional strategy at all levels of teaching. Dwyer (1988) points out numerous justifications that support the use of visualisation in the instructional process. These include: the increase of learner interest and concentration; the isolation of specific instructional characteristics; the introduction and presentation of new information; the illustration, emphasis and reinforcement of both oral and printed forms of instruction; the development of discrimination, identification, clarity of thought and the ability to draw conclusions.

However, according to Dwyer (1988: 367) research shows that there are often no significant differences in learning occurring by means of visually mediated instruction as opposed to conventional types of teaching and the various explanations which have been offered for the apparent lack of increased student learning resulting from visualisation instruction. These include: the loss of information which takes place during the transformation of energy from the eye to the brain; the verbal loop hypothesis, contending that visual information is translated and stored in verbal or symbolic form and, when it is retrieved, is translated back from the verbal symbolic to the original visualisation.

It seems that although visual stimuli are important, too much reliance on imagery in mathematics can limit mathematical performance and give rise to difficulties. Although the

use of concrete materials may be beneficial to some students, it may not be assumed that all students have the interpretive skills to relate visually presented items to the underlying mathematical entities. Nevertheless, students seem to benefit from forming mental pictures in mathematics and integrating analytic and visual thinking.

3.5.5 Visualisation and problem solving

Cangelosi (1996) has a nine-stage strategy for solving problems. His fourth stage is described in the following manner "The situation is visualized so that relevant relationships involving the principal variable or variables are identified and possible solution designs are considered" (Cangelosi 1996: 51). Visualisation also plays a significant role in his eighth stage when "Results of the executions of processes, formulas, or algorithms are interpreted to shed light on the original question or questions"

Sawyer (1964: 8) believes that it is possible for results in arithmetic to be understood "by a single act of mental vision". Although small numbers may be easily visualised, he comments that "How to organise the chaos that lies beyond the smallest numbers is therefore a problem that confronts the entire human race" (Sawyer 1964: 8). He is in favour of encouraging children to get into the habit of thinking as they are introduced to new topics so that they are made aware that they can solve problems for themselves. The approach followed by the teacher is very important because "The difficulty of learning a subject depends enormously on the way in which the subject is presented" (Sawyer 1964: 1).

In the lessons to follow on sequences and series, problem solving techniques will be used and students will be provided with visual examples in order to establish and reinforce concepts. For example, students will be provided with numerous examples of sequences, arithmetic sequences and geometric sequences in order to establish their definitions. Alfred Bennet's visual-geometric representation of geometric series will be used. He believes that geometric models help to provide understanding and insight regarding theorems and their proofs. These do help to show the interplay between both algebraic and geometric

approaches and give students a chance of discovering and understanding mathematics for themselves. Another advantage of models is that they serve to stimulate discussion within a group. Hurwitz (1993: 37) observes that "The abstract nature of algebra makes manipulative experiments, visualization activities, and motivational introductions, reviews, and extensions all the more valuable" She too provides a visual approach for summing terms of an arithmetic series which will be followed in the lessons of Chapter 4.

3.5.6 Conclusion

Visualization has been of major significance in mathematics for many centuries. Ancient Greek Mathematicians used to sketch diagrams in the sand and today many people such as mathematicians, scientists, architects, and artists engage in visual activities. Visual comprehension plays a significant role in school mathematics too. Students need to be encouraged to recognise a picture; find the properties, similarities and differences in a representation; recognise objects that have been turned around as well as embedded figures; imagine or visualise "... a situation from a written or oral description" (Davey & Holliday 1992: 26).

Thus students should be encouraged to make use of visualisation in their everyday mathematical thinking. It is important that visualisation be effectively integrated into the teaching/learning situation. Piaget, van Hiele and Freudenthal, are in favour of the idea. However, teachers need to consider carefully whether visual materials do effectively complement their instruction. In addition, they need to determine whether their visual illustrations are suitable for the different comprehension levels of their pupils. It should be ensured that students actually do relate visual illustrations to the underlying concepts and perceive the relevant cognitive links in the teaching/learning process.

3.6 Patterning and generalisation

3.6.1 Processes view of mathematics

There are different views regarding the nature of mathematics and how children learn mathematics. Skemp (1971) draws a distinction between instrumental and relational understanding which is reflected in the formalistic and processes approaches. The formalistic view is described by Christiansen as "... a set of concepts, rules, theorems and structures. To acquire this body of knowledge you need certain algorithmic skills and deductive reasoning ability" (Moodley 1992: 4). Another view is that mathematics consists of processes like classifying, ordering, exploring patterns, formalizing, abstracting, generalizing and so on.

The two abovementioned views have different implications for mathematics teaching. The former view is concerned with the finished product of mathematical activities, emphasizes procedures, manipulative techniques and routine problems. In most cases the teacher tells and shows the students what to do, continually repeating what has been said. The latter view results in mathematics being learnt as a process regardless of the content. The emphasis is mainly on the development of meaningful concepts and generalisations with more prospects for open enquiry, investigation and real problem solving. Here the teacher tends to question, challenge and guide students by encouraging them to actively discover and apply what they have learnt in the classroom. The two processes of exploring patterns and generalising will be emphasised here.

3.6.2 Exploring patterns

3.6.2.1 Introduction

The word pattern is difficult to define because it has such a wide variety of meanings. Orton (1999: vii) points out.

On the one hand, 'pattern' can be used simply in relation to a particular disposition or arrangement of shapes, colours or sounds with no obvious regularity. Indeed, sometimes the arrangement might form a recognisable representation or picture. On the other hand, it might be required that the arrangement possesses some kind of clear regularity, perhaps through symmetry or repetition. In mathematics, we more often than not use the word 'pattern' in relation to a search for order, so regularity is more likely than not.

Many mathematicians and educationists have expressed enthusiasm about the importance of pattern in mathematics. For example, Sawyer (1955: 12) stated that "... mathematics is the classification and study of all possible patterns" Orton (1999: vii) and Biggs and Shaw (1985: 1) remark that it is possible to think of mathematics as being "... a search for patterns and relationships". The Pattern in Mathematics Research Group was set up at the School of Education of the University of Leeds in early 1992 in order "...to provide structure and support to our developing studies of children's perception, conceptions and use of pattern in learning mathematics" (Orton 1999:vii).

3.6.2.2 Exploring patterns in the teaching and learning of mathematics.

Exploring patterns usually helps pupils extract greater meaning from their learning environment and facilitates remembering too.

According to Gestalt psychology (Orton 1999: vii),

... it is a human quality to want to interpret incoming sensations and experiences as an organised whole rather than a collection of separate pieces of data. This reflects the need to encourage the use, perception and comprehension of patterns wherever possible in the teaching of mathematics.

Arons holds that there exist a number of basic patterns or processes of thinking and reasoning which underlie almost all learning and understanding. He states:

It is my conviction that helping students become explicitly conscious of these patterns, and giving them repeated opportunity to practise and exercise such modes of thought in successive, different contexts of subject matter, greatly enhances their grasp of concepts and principles as indicated by gradually improving ability to analyze physical phenomena and to make predictions in new or altered situations (Arons 1983: 516).

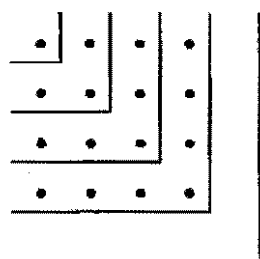
He suggests that a powerful way of helping students master a mode of reasoning is to allow them to view the same reasoning from more than one perspective.

Patterning itself is a form of problem solving which can be enjoyable and challenging to students. It enables them to approach a new task by thinking about it rather than asking how to do it. As patterns are repeated, pupils are confirming for themselves the pattern solved or discovered earlier. They are often anxious to share their ideas and add to the list of

observations made in a classroom situation. The discussion phase of patterning is justified as it leads to verbalisation, an important part of problem solving.

3.6.2.3 Patterns in the teaching and learning of sequences and series.

Many countries begin teaching odd and even numbers in the primary school phase. There are many interesting patterns to be found in relation to these numbers. For example, as mentioned earlier, the sum of the odd numbers starting with 1 is always a square number as illustrated by the pattern below:



$$1 + 3 + 5 + 7 = 4^2$$

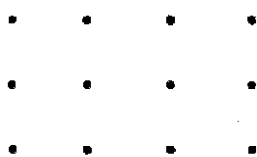
$$\text{and in general, } 1 + 3 + 5 + \dots + (2n-1) = n^2$$

figure 9

Further, the sum of the even numbers has a non-square rectangular pattern as follows:



$$2 + 4 = 6 = 2 \times 3 \quad \text{figure 10}$$



$$2 + 4 + 6 = 12 = 3 \times 4 \quad \text{figure 11}$$

In general, $2 + 4 + 6 + \dots + 2n = n(n+1)$

Orton refers both to the suggestion of Bruner that learning generally proceeds from the practical to the pictorial to the symbolic and to Mason that the first stage of learning is seeing, such as seeing the dot pattern of triangular numbers below:

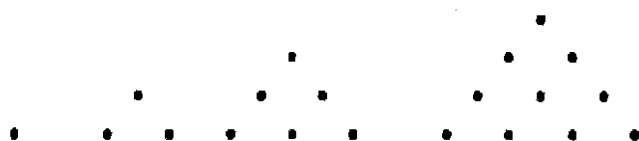


figure 12. (Orton 1999: 126).

This does have the advantage that concrete or pictorial representations are perceived to be

more simple than symbolic ones. However, it could be argued that numerical symbols are concrete enough for secondary school students. Pictorial illustrations may lead to different numerical approaches being followed by students. For instance, they could count the dots and use the results to extend their sequences. They could also focus on the differences between sequences or imagine subsequent pictorial representations in their minds. Orton (1999) noted how, although some children prefer practical contexts whereas others prefer analytical or verbal approaches, most children do seem to benefit from a combination of both approaches.

It has been found that pictorial representations are not always linked to their corresponding numerical tables of values so that it is a good idea to represent the two forms of depicting a pattern together. For example, in the below table corresponding to the given pattern, the number denoting the position of a term in a sequence is represented together with the term


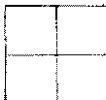

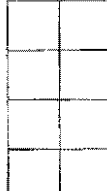
				
Term number n	1	2	3	4
Term T_n	2	4	6	?

figure 13

Varieties of types of sequences need to be provided at an initial stage so that pupils do not merely consider the difference between terms. Pupils are accustomed to regular spacing in number sequences. For example 1; 1; 2; 3; 5; 8; 13; makes the Fibonacci sequence pattern difficult to detect and Pascal's triangle is not as effective when the first five rows are written in the following manner:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	

figure 14

3.6.2.4 Conclusion

Exploring patterns helps to make pupils more confident to persist in searching for patterns within data. Patterns can be explored, extended, created and generalised initially making use, if at all possible, of a whole variety of materials. In order to introduce students to new concepts at different levels in the learning of sequences and series, patterns could be utilised to promote understanding and give a mixed approach involving both pictorial and analytical representations.

3.6.3 Generalisation

3.6.3.1 Introduction

Once patterns have been established, the goal is to move beyond to a further level involving generalising the findings made. Generalisation refers to going beyond what is explicitly given. Different kinds of generalisation may be recognised. This is because it is possible to go beyond what is given in different ways and to different extents.

For example, considering the sequence 1; 3; 5; 7, the fact that terms increase by 2 makes it easy to work with and continue and also gives understanding regarding the formation of odd numbers. However, actually knowing that the numbers of the sequence are odd is a generalisation about a property of a set of numbers. A more sophisticated mathematical generalisation would be symbolising the rule in algebraic terms by representing the n_{th} term of the sequence by $T_n = 2n - 1$. This allows any term of the sequence to be predicted. For example, the twentieth term would be $2(20) - 1 = 40 - 1 = 39$.

By observing the arithmetic and geometric sequences, the formulae $T_n = a + (n - 1)d$ for arithmetic sequences and $T_n = ar^{n-1}$ for geometric sequences could be established. In addition, deriving formulae for sums of terms in sequences and series will form an important part of the subsequent lessons on sequences and series.

3.6.3.2 Generalisation in algebra

Dörfler's generalisation model of 1991 regards generalisation as being the essence of mathematical conceptualisations and gives a description of the generalisation process. The pre-stage here is called constructive abstraction in which reflection on objectives and methods leads to the abstraction and relation of elements. Extensive generalisation refers to the applicability and relations of these properties. However, symbolic description removes the object from the original context to the symbol, leading to intensional generalisation. (Iwasaki, Yamaguchi & Tagashira 1998) refer to his model as unique because it includes the symbol and refers to the role of the symbol, thereby making it have implications for mathematics teaching.

Orton (1999) claims it is possible to introduce concepts in algebra through generalisation from number patterns and that findings indicate that the route through number patterns does not remove all difficulties concerning beginning algebra but remains a justifiable route if its limitations are accepted and problems are circumvented. He points out that there could be a danger in always ignoring pictures and immediately finding patterns from a corresponding list of numbers. An example of Mason et al illustrating the importance of the original context involves a question in which a rectangle 2 squares wide and 3 squares long is presented as follows:

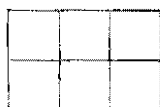


figure 15

The following question is asked: "I want to put a border one square thick all the way round.

How many squares will be needed for other rectangles" Orton (1999: 124).

Different ways of arriving at the solution $2L + 2B + 4$ included: $L + B + L + B + 4$ (for the corners); $2(L + B) + 2(B + 2) - 4$. (to make up for counting the corners twice); $(L + 2)(B + 2) - LB$ (by subtracting an inside rectangle from a larger rectangle). None of these approaches depends on collecting and tabulating large quantities of data (though this might have been valuable in providing ideas). Clearly in this instance the crucial insights come from relating these data to the original context.

3.6.3.3 Generalisation in sequences and series

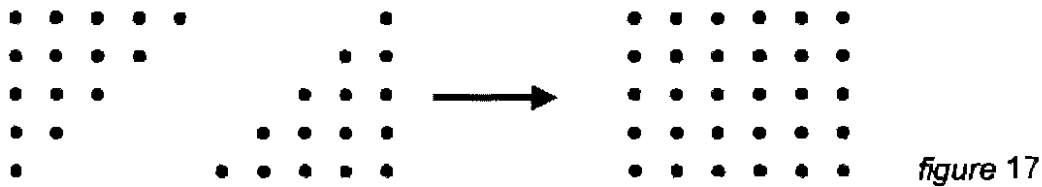
It has been found that although some children can use generalised methods for finding the terms of a given sequence, their generalising ability does not extend to subsequent sequences. This would seem to suggest the provision of a variety of different types of examples in various contexts. There are times when pictures do not appear to provide direct help in generalisation. Enough attention needs to be given to working with both visual and algebraic or verbal systems in parallel so that pupils grow accustomed to moving from one system to another.

Below is an interesting example showing how visual images can lead to generalisation in sequences and series.



figure 16 (Orton 1999: 126).

The pattern is not immediately recognisable by differencing since the sequence 1;3;6;10 does not have an obvious general term although the numbers are clearly increasing by 2,3,4 and so on. Differencing does show a recursive understanding of the pattern, but in this case it is not an adequate kind of understanding and possibly even misleading in attempting to find the n_{th} term. However, arranging them twice as follows gives us:



(Orton 1999: 127)

The number of dots in the rectangular array on the right hand side is clearly

$$\begin{aligned} 2 \times (1 + 2 + 3 + 4 + 5) &= 30 \\ &= 5 \times 6 \end{aligned}$$

and so the required formula is $S_n = \frac{1}{2} \times n \times (n + 1)$.

Without being taught this approach, pupils are not very likely to discover it for themselves.

Once they have learnt the formula for summation of terms of an arithmetic series, recognising that the triangle numbers are related to the single rule growth $1 + 2 + 3 + 4 + \dots$, since this is an arithmetic series with $a = 1$ and $d = 1$, the n_{th} sum is given by the formula

$$S_n = \frac{n}{2} [2a + (n - 1)d]$$

Orton (1999) describes a possible strategy for generalising about sequences. In it, all relationships between adjacent terms are observed and then common results are considered in order to lead to generalisation. Once combined to form a strategy to attain a goal, processes may be stored in a child's long term memory, especially if the strategy has been used in a range of situations. After being retrieved from long term memory as a collection in a given order, these processes may be used once again to achieve similar goals. A planning stage needs to precede the implementation of a strategy because difficulties can arise when inappropriate strategy choices are made or when a strategy is applied incorrectly.

3.6.3.4 Conclusion

It seems that in order to promote better strategy use and improved generalising, pupils should be provided with sequences covering a wide range of structures. It is important to provide tasks which enable them to work with and appreciate sequences in their own right. A wide range of pattern structures and tasks help children to become more persistent in searching for patterns within data, less reliant on strategies looking for a difference between terms and more able to use more than one strategy or choose information in an appropriate manner to form a generalisation. Matchsticks help to provide another way of both teaching and researching the use of patterning leading to generalisation in sequences and series. They also provide the opportunity of seeing a sequence in different ways and this will form part of one of the initial activities in Chapter 4.

It has been found that mature adults too are more likely to base their attempts on differences and have great difficulty in dropping this approach and seeking a better one which will lead to a general rule. Since patterning and generalisations are very important in the study of mathematics, the development of strategic thinking and generalisation regarding sequences and series helps to form a solid mathematical foundation.

3.7 Summation

The learning theories of Piaget, van Hiele and Freudenthal all indicate the existence of levels in the development of mathematical thinking. Since visualisation, patterning and generalisation all seem to have a significant role in the progress through the various thought levels in the learning of mathematics, their impact on advancement through the thought levels involved in the learning of sequences and series will be considered in subsequent chapters.

CHAPTER 4

Lessons on sequences and series

4.1 Introduction

The theories of Piaget, van Hiele and Freudenthal all indicate levels of learning in mathematics. In the following lessons on sequences and series, the four previously mentioned learning levels of Land known as the base level or pre-descriptive level called level 0, the descriptive level or level 1, the theoretical informal level or level 2 and theoretical formal level or level 3 will be considered. Lessons are designed to take students through these levels as they progress through various topics. The role of visualisation and the parts played by the processes of patterning and generalisation will be considered in the advancement from one level to the next. Proof by mathematical induction will be included at the end of the series of lessons as a level three (or fourth level) activity.

The extracts of lessons and questionnaires presented here all form part of a series of lessons on sequences and series presented to six higher grade mathematics students from a local secondary school. There were three girls and three boys and the six students were divided into two groups of three students each. Initially the students formed their own groups but after a few of the fourteen lessons they were rearranged into mixed ability groups with two girls and one boy in the one group and two boys and one girl in the other. All of the lessons and questionnaires are contained in the appendices A and B respectively.

4.2 Van Hiele levels in sequences and series

As has been mentioned in Chapter 3, in agreement with van Hiele's suggestion and the approach followed by Land, a presdescriptive level called level 0 will be used here which is not a true van Hiele level. Numerous patterns and visual representations will be provided for students at this level in order to enable them to recognise a sequence when they see one in various contexts.

At the second (descriptive) level or level one the objects will be the properties of sequences and series which will be established in an inductive manner. At this stage the student recognises the algebraic properties of a sequence or series, is able to state them accurately, can formulate expressions involving symbols and perform elementary manipulation of symbols.

The third theoretical level or level two concerns "... empirically established relationships between the properties" Land (1990: 133). The students are able to give concise and accurate definitions and follow a deductive argument. In addition, they are able to follow the derivation of such formulae as

$$S_n = \frac{n}{2}[2a + (n-1)d] \text{ and } S_n = \frac{a(r^n-1)}{r-1} \text{ or } S_n = \frac{a(1-r^n)}{1-r}$$

Pupils are able to utilise these formulae in various contexts in order to solve problems.

At the fourth or theoretical level known as level three, the subject matter is "... partial orderings of statements that relate the properties" Land (1990: 134). Further, she quotes a letter written by Fuys which indicates the difference between levels two and three.

I interpret level three as meaning 'logically established relationships not those which are formulated on the basis of experimentation or inductivity. I have spoken with Pierre van Hiele about this, and I believe we agree. Children can discover relationships about shapes or functions by working with examples, but the relationship is not logical (if... then where the implication is by deduction). However, if a student uses a definition and argues that something must be so then the student is establishing the relationship logically. (Land 1990: 133)

Various types of level three activities will be included here. The students will deduce the formula $S_\infty = \frac{a}{1-r}$ from the formula $S_n = \frac{a(1-r^n)}{1-r}$. They will be required to solve more complex problems than those on level two and develop meaningful insight with regard to symbols. Here such examples as word problems regarding sequences and series as well as questions such as the following ones will be asked:

- What is the greatest value of m for which $\sum_{k=1}^m 7(3)^{k-1} > 10^6$?
- If k is an even number, find the sum of the series:

$$\frac{1}{k} + \frac{3}{k} + \frac{5}{k} + \dots + \frac{k-1}{k}$$

Hence or otherwise, evaluate:

$$\left(\frac{1}{4} + \frac{3}{4}\right) + \left(\frac{1}{6} + \frac{3}{6} + \frac{5}{6}\right) + \dots + \left(\frac{1}{50} + \frac{3}{50} + \dots + \frac{49}{50}\right)$$

A very important level three activity here will be proof by mathematical induction. The students will be led gradually to this level by being encouraged to appreciate many number patterns and the need for this type of proof. They should be able to recognise situations in which this mode of proof is applicable.

4.2.1 1st level - level 0 - the pre-descriptive level.

In agreement with the description by Land (1990), the following objects and characteristics will be recognised at this level:.

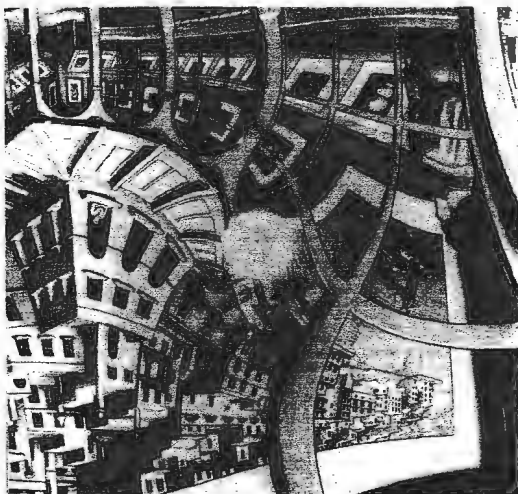
Objects : Sequences and Series.

Characteristics :

- Recognising a sequence or series in different situations.
- Recognising a particular type of sequence or series in various contexts.
- Associating the correct name with a sequence, series or a particular type of sequence or series.

Below follows parts of worksheets used in lessons to illustrate the type of activities performed at this level. All the worksheets provided are included in the appendix.

First of all, patterns were studied and the following extracts from Worksheet 1.1 indicate how patterns and visual examples are provided to establish the concept of a sequence.



Escher in (Serra 1997: 1)

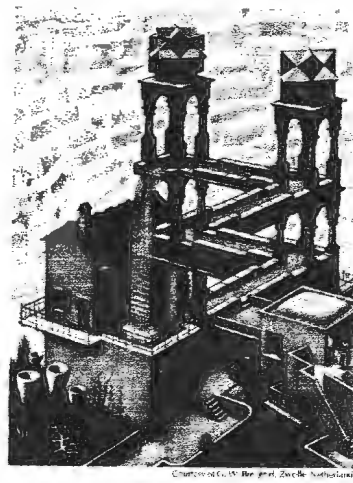


figure 18.

Escher in (Jacobs 1970: 19)

1. What is a pattern?
2. Give 2 examples of patterns.

In Worksheet 1.2 and 1.3 pupils were required to make matchstick patterns, repeat patterns on a grid, complete patterns of the following types.

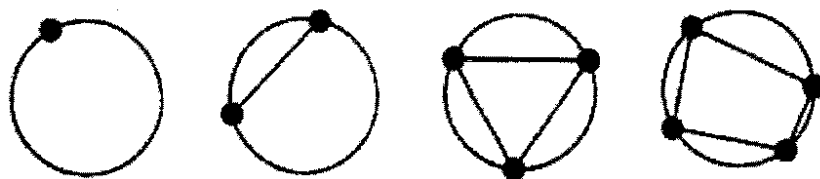


figure 19.

Serra (1997 : 46)

The following Worksheet 2.1 was used to lead to the concept of what a sequence actually is:

1.

Take a fairly large sheet of notebook paper and fold it in half, making a sharp crease. Turn the paper and fold it in half again at right angles to the first fold, again making a sharp crease.

You have now folded the paper twice.

If the paper has been folded seven times, how many thicknesses of paper are in the wad produced?

To find out, we can make a list of the number of thicknesses of each wad starting with the first one.

No. of folds	:	1	2	3	4	5	6	7
--------------	---	---	---	---	---	---	---	---

No. of thicknesses	:	2	4	8	—	—	—	—
--------------------	---	---	---	---	---	---	---	---

2.

Toss a coin. If you get heads write a 1 beneath the number of the throw and if you get tails write a 0 under the number of the throw as follows:-

No. of throw	:	1	2	3	4	5	6
--------------	---	---	---	---	---	---	---

Result (heads or tails)	:	0	0	1	?	?	?
-------------------------	---	---	---	---	---	---	---

Draw up your own chart in this way. After tossing the coin three times, can you guess what your fourth, fifth and sixth result will be?

3.

There is a legend that the King of Persia offered the inventor of the game of chess anything he wanted as a reward. What the inventor requested did not seem like much. He asked that one grain of wheat be placed on the first square, two grains on the second square, four grains on the third and so on, each square having twice the number of grains as the square before. The King thought that this was a reasonable request so he sent a servant for a sack of wheat. Can you guess what happened?



figure 20 (Jacobs 1970: 54)

4.

Suppose that 27 members of a matric class are present at school on the first day and 30 on the second day, 23 on the third day and 29 on the fifth day. Have you any way of knowing how many pupils were present on the fourth day? Explain your answer.

Day	:	1	2	3	4	5
No. of Pupils	:	27	30	23	?	29

5.

Any ordered list such as 2; 4; 6; is called a sequence. A Fibonacci sequence, named after Leonardo Fibonacci, is illustrated by the arrangement of leaves and flowers on the plant below. Try to detect the pattern and then write down the first 10 terms of the sequence.



(Jacobs 1970: 76)



Leonardo Fibonacci

(Jacobs 1970: 72)

figure 21

figure 22

In Worksheet 3.1 arithmetic sequences were introduced at level 0 by providing students with both examples and non examples. Questions were then asked to encourage the students to generalise and advance to level 1 type thinking. Geometric sequences were introduced in a similar manner to arithmetic sequences as illustrated by extracts from Worksheet 4.1 below.

E4 3; 6; 12;

N4 3; 6; 9;

E6 1; -1; 1; -1;

N6 1; -1; -1; 1;

- E7 The sequence of the heights of the bounces of a ball dropped from a height of 1 metre and bouncing up exactly one half the distance it has just come down.
- N7 The list of the number of pupils of a class of 30 (taken in alphabetical order), who were present each day of the second term this year.

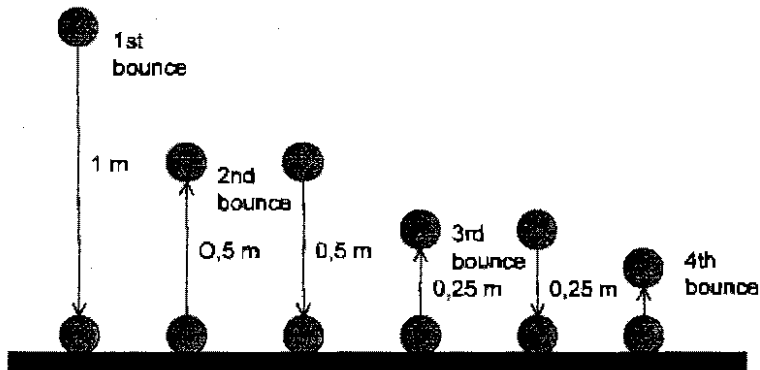


figure 23

E11 0,7; 0,07; 0,007; N11 0,7; 0,77; 0,777;

E12 $3x; 3x^2y^2; 3x^3y^4; \dots$ N12 $3xy; 6xy; 9xy; \dots$

- 2.1 How are the examples alike?
- 2.2 How are the non-examples alike?
- 2.3 How do the examples and non-examples differ?
- 2.4 How do the examples differ from each other?

The concept of a series was also introduced at the pre-descriptive level. At this stage the concept of a sequence had become prior knowledge. The approach is illustrated below by an extract from Worksheet 5.1.

1. Consider the following sequences of squares with sides of length 1cm, 2cm, 3cm and 4cm.

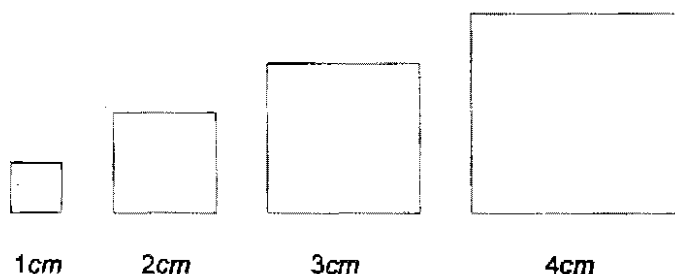


figure 24

- 1.1 Suppose we were to construct squares of these sizes using a thin piece of wire. State how much wire would be needed for the

first square _____

second square _____

third square _____

fourth square _____

After this, T_1 , T_2 , T_3 , and T_4 were used to represent the lengths of wire required to make the squares and S_1 , S_2 , S_3 and S_4 were used to represent the amount of wire used to represent 1, 2, 3 and 4 squares.

The sort of examples used above are simple ones which can be understood by being pointed out as mentioned by van Hiele (1989) in his correspondence with Land. They are designed to facilitate progress to level 1 type thinking. This pre-descriptive level has been described by Fuys, Giddes and Tischler as being "... analogous to a ground floor of a building - it represents the type of thinking that all students will initially bring to a new subject" (Land 1990 : 132).

4.2.2 2nd level - level 1 - the descriptive level.

Objects and characteristics similar to those described by Land (1990) regarding the learning of functions will be associated with this level.

Objects - properties of sequences or series which may be established in an inductive manner.

Characteristics

- Recognising and accurately stating algebraic properties of sequences and series

- Discovering formulae by experimentation.
- Formulating expressions involving symbols and performing elementary manipulation of symbols.
- Recognising that a change in formula represents a change in the sequence without the explicit use of pictures or particular examples.

The following are extracts from Worksheet 2.3 intended for level one.

1.2

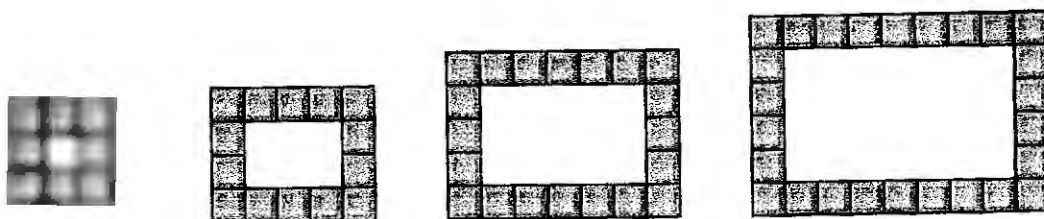


figure 25.

(Serra 1997: 72)

Rectangular pattern

Term number	1	2	3	4	5	6	n	100
Number of squares										

2.

If the pattern of rectangles were to continue, what would the rule be for the number of squares in the n th rectangle?

What would the number of squares in the 200th rectangle be?

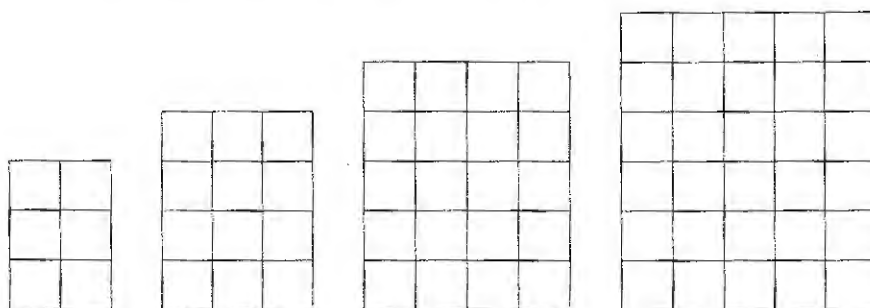


figure 26

Squares in a rectangular array

Rectangle	1	2	3	4	5	6	...	n	...	200
Number of squares										

Extracts from question 2 of Worksheet 2.4 follow below:

2. Find the values of T_4 ; T_n ; T_{100} in each of the following sequences:

	T_4	T_n	T_{100}
2.3 -7 -14 -21			
2.5 10 7 4			

In Worksheet 3.3 students were led to the discovery of the formula T_n for the n th term of an arithmetic sequence and were given opportunities of utilising this formula in various simple examples.

3. Suppose that a is the first term and d is the common difference of an arithmetic sequence.

1st term

a

2nd term

a	d
-----	-----

3rd term

a	d	d
-----	-----	-----

4th term

a	d	d	d
-----	-----	-----	-----

5th term

a	d	d	d	d
-----	-----	-----	-----	-----

(Jacobs 1994: 65)

- 3.1 List T_1 , T_2 , T_3 , T_4 , T_5 and T_6 .

- 3.2 If k is any natural number, find an expression for T_k .

- 3.3 What is T_{1000} ?

More questions follow from Worksheet 3.4 include:

Write down the formula you have derived in worksheet 3.2 for the general term T_k of an arithmetic sequence with first term a and common difference d .

--

1. List the first three terms of an arithmetic sequence with

1.4 $a = -\frac{3}{4}; d = \frac{1}{2}.$

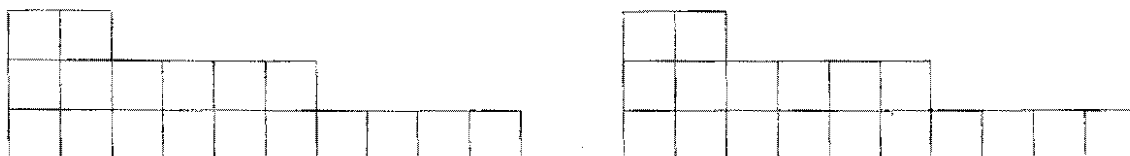
1.5 $a = 2x + y; d = -x - y.$

Similarly, after being introduced to the concept of the geometric sequence, the students were encouraged to discover the general formula for the n th term of a geometric sequence. This was followed by various simple applications. Later, in the initial lesson on arithmetic series, on Worksheet 6.1 students were provided with block patterns to cut out and arrange in order to encourage them to find a general rule for summing terms of arithmetic series and hence promote their advancement to the derivation of the formula at level 2.

1. Consider the series

$$2 + 6 + 10$$

Illustrated twice below: figure 27



- 1.1 Cut out both the block patterns and combine them to create a rectangle.
 1.2 What is the width of the rectangle?
 1.3 What is the length of the rectangle?
 1.4 What numerical characteristic of the series is equal in value to the width of the rectangle?
 1.5 What numerical characteristic of the series is equal in value to the length of the rectangle?
 1.6 How does the sum of the series $2 + 6 + 10$ compare with the value of the area of the rectangle?

2. Repeat the same procedure with the series $3 + 7 + 11 + 15$

and $2 + 4 + 6 + 8 + 10$ and then complete the summary sheet below:

(The summary sheet is contained in Appendix A)

Students were led to a formula for summing up a term of geometric series to n terms in Worksheet 7.1. Studying the pattern to be found in Pascal's triangle in Worksheet 9.1, gave them a further opportunity of rising from level 0 to 1. Results were generalised and used to

expand powers of binomials. In these worksheets students were provided with opportunities of visualisation and appreciating patterns. They were also encouraged to generalise and learn relevant terminologies so that they would be prepared to move up to the next level of thought in each respective topic.

4.2.3 3rd level - level 2 - the informal theoretical level.

Taking into consideration the third algebra level mentioned by Land (1990), the following objects and characteristics would seem to be appropriate for level 2 in the teaching of sequences and series:

Objects - Statements showing relationships between the properties of sequences and series.

Characteristics -

- Recognising interrelationships between different type of sequences and series.
- Defining words accurately and precisely.
- Understanding statements that relate properties of sequences and series.
- Solving equations involving manipulation of symbols.
- Formulating statements showing interrelationships between symbols.
- Following a derivation of a formula.
- Following a deductive argument.

The following extracts reveal level 2 type activities. Visualisation, patterning and generalisation are encouraged to help pupils master this level and prepare themselves to advance to the fourth level.

Students were provided examples of solving equations involving manipulation of symbols and statements involving properties of arithmetic sequences in Worksheet 3.4 as illustrated below:

6. Given the arithmetic sequence

$$-4\frac{1}{4}, -3\frac{1}{2}, -2\frac{3}{4}, \dots$$

6.1 Which term in the sequence is 13?

6.2 Calculate T_{13} .

7. If $3y - 1$, $-2y + 3$ and $2y - 1$ are the first three terms of an arithmetic sequence

7.1 Calculate y .

7.2 Determine the first 3 terms of the sequence.

7.3 Determine a formula for the k th term.

7.4 Which term in the sequence is -61 ?

7.5 Calculate the 19th term.

Students were led to the derivation of the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$ for the sum to n terms of an arithmetic series in Worksheets 6.2, 6.3 and 6.4.

For example in Worksheet 6.3 the following questions were asked:

1. Consider the sum of the first hundred numbers below:

$$S_{100} = 1 + 2 + 3 + \dots + 98 + 99 + 100$$

In the space provided below, rewrite $S_{100} = 1 + 2 + 3 + \dots + 98 + 99 + 100$ again but with the terms in reverse order

$$\text{i.e. } S_{100} = 100 + 99 + 98 + \dots + 3 + 2 + 1.$$

Then carefully consider the patterns you see and consider how you could find S_{100} from this representation.

6.1 Find the sum of S_{10} for the series $a + (a + d) + (a + 2d) + \dots +$

6.2 Find the sum of S_n for the series $a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$

$$\text{Let } l = T_n = a + (n - 1)d$$

Similarly, students were led to a formula for summing up the terms of geometric series in Worksheets 7.3.

1. In the geometric series $1 + 3 + 9 + \dots$ of the previous worksheet, what are a and r ?

2. Write down S_7 and rS_7 where rS_7 is the series obtained by multiplying each term of the series S_7 by r .

3. What pattern do you notice if rS_7 is written below S_7 with like terms underneath each other?

4. How could you use what you have observed in 3 to help you find S_7 without adding up term by term?

5. Check your answer obtained in 4 on your calculator.

Later came the following questions 5 and 6 in Worksheet 7.4.

5. Derive a formula representing S_n for the sum to n terms of the general geometric series.

$$a + ar + ar^2 + \dots + ar^{n-1}$$

6. Could you think of another way of writing down the formula you have derived in 5?

Students were subsequently provided with examples of solving equations involving manipulation of symbols in Worksheet 7.4. In Worksheet 9.2 the students were provided with various further number patterns in order to encourage them to make deductions and build a firm foundation for mathematical induction.

For example,

2. There are many patterns which arise from sequences of numbers.

- 2.1 For example, consider the sum of consecutive odd numbers, as follows:

$$1 =$$

$$1 + 3 =$$

$$1 + 3 + 5 =$$

$$1 + 3 + 5 + 7 =$$

$$1 + \dots$$

$$1 + \dots$$

- 2.2 What do you think the rule is here?

- 2.3 Complete the following diagram and see whether it illustrates your results in 2.1 and 2.2.

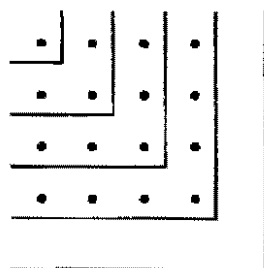


figure 28

- 2.4 Do you think this result has been proved or needs to be proved?

The following example was also provided to show students that since it is possible for a statement to be true for 40 values of n but not 41, proof regarding apparent number patterns is necessary.

4. Consider the sequence with $T_n = n^2 - n + 41$. We wish to investigate whether every term will be a prime number.

4.1 Find the first ten terms. i.e. Complete the following table:

n	$n^2 - n + 41$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

(Fleming & Varberg 1989: 427)

4.2 Are all your answers prime? (A prime number is a number with exactly two factors, itself and one.)

4.3 Is T_n a prime number for all n ?

4.4 Find the value of T_{41} .

4.5 Do you need to prove a result or can we conclude it must be true if it is true for the first ten or so values? Explain.

The above two examples were designed to prepare pupils for the need for proof by mathematical induction to be introduced on the next level.

4.2.4 4th level - level 3 - the formal theoretical level.

The Objects and Characteristics associated with level 3 here are similar to those mentioned by Land (1990) in connection with the teaching of algebraic topic of functions. These include;

Objects - Partial ordering of statements related to properties of sequences and series.

Characteristics -

- Using information about sequences and series to deduce more information.
- Understanding the significance of deduction.
- Using definitions to establish arguments.

- Understanding the role played by postulates, theorems and proofs.
- Using symbols with insight and understanding to construct a proof or solve a problem.
- Formulate arguments based on diagrams or visual representations.
- Being able to verbally summarise an argument, identify, hypothesis, intermediate deduction and conclusion.

In the following Worksheet 8.1, students are encouraged to formulate arguments based on computer generated figures, diagrams or visual representations.

3. Recall the sequence of numbers

1; 1; 2; 3; 5; 8; 13; 21; 34; —————

3.1 What is this sequence called?

3.2 Calculate the first thirty terms of the sequence.

3.3 Evaluate $\frac{T_2}{T_1}, \frac{T_3}{T_2}, \frac{T_4}{T_3}, \dots, \frac{T_{k+1}}{T_k}, \dots$

i.e. $T_2 \div T_1; T_3 \div T_2; T_4 \div T_3, \dots, T_{k+1} \div T_k, \dots$

3.4 Do you think $\frac{T_{k+1}}{T_k}$ tends to a limit as $k \rightarrow \infty$?

3.5 Evaluate $\frac{\sqrt{5}+1}{2}$.

3.6 What do you notice?

The number $\Phi = \frac{\sqrt{5}+1}{2}$ is called the Golden Ratio and has a value of approximately 1,618 when expressed in decimal form.

In Worksheet 8.2 the following visual example was provided.

1. We shall first consider geometric series for which $a = 1$ and beginning with the second term.

i.e. $a + ar + ar^2 + ar^3 + \dots$

becomes $r + r^2 + r^3 + \dots$

If we let $r = \frac{1}{k}$ where k is a whole number greater than 1, then this becomes

$\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \dots$

If $k = 2$, then we get

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

Now consider the sum of the first few terms of the series as follows:

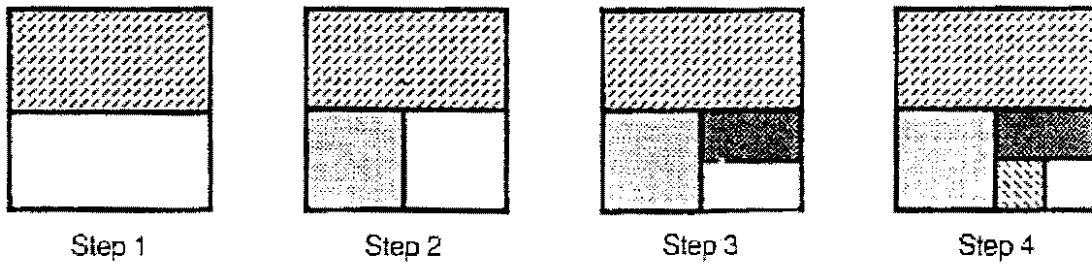


figure 29

(Bennet 1989: 134)

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

1.1 What do you think will be the value of $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$?

In Worksheet 8.3 graphical illustrations of limits were given:

2. Now consider the following cases of geometric series:

2.1 $|r| < 1$ i.e. $-1 < r < 1$

e.g. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$S_1 =$ $S_2 =$ $S_3 =$ $S_4 =$ $S_5 =$

Plot the values on the graph below:

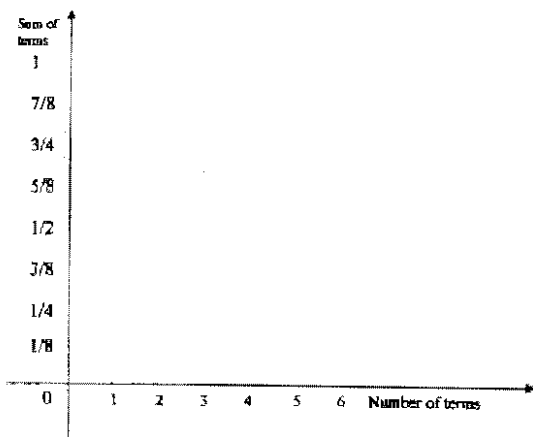


figure 30a (Laridon 1996: 89)

We can conclude that:

as $n \rightarrow \infty$, $S_n \rightarrow$ _____

i.e. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n =$ _____

2.2 $r > 1$ e.g. $1 + 3 + 9 + \dots$

$$S_1 = \quad S_2 = \quad S_3 = \quad S_4 = \quad S_5 =$$

Plot the values on the graph below:

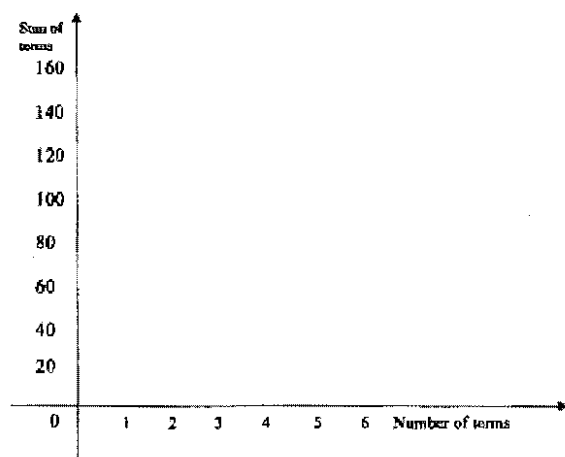


figure 30b (Laridon 1996: 90)

What do you think happens as $n \rightarrow \infty$?i.e. as $n \rightarrow \infty$, $S_n \rightarrow$ _____

Students were then led to deduce the formula $S_\infty = \frac{a}{1-r}$ and applied it in various situations.

They also solved a variety of mixed problems involving all theory they had learnt regarding sequences and series in Worksheet 8.4.

Consider the formula

$$\begin{aligned}
 S_n &= \frac{a(1-r^n)}{1-r} \\
 &= \frac{a-ar^n}{1-r} \\
 &= \frac{a}{1-r} - \frac{ar^n}{1-r} \\
 &= \frac{a}{1-r} - \frac{a}{1-r} \cdot \frac{r^n}{1}
 \end{aligned}$$

Suppose $|r| < 1$ i.e. $-1 < r < 1$.e.g. $r = \pm \frac{1}{2}$, etcWhat will happen to the value of r^n as $n \rightarrow \infty$?Hence can you find a formula for S_∞ ?

$S_\infty =$

Does this formula hold only for $|r| < 1$?

4. For which values of x will the infinite series $1 + \frac{1}{2x-1} + \frac{1}{(2x-1)^2} + \dots$ converge?

In Worksheet 9.3 students were made aware of the truth or falsity of statements such as the following.

$P_n : n^2 - n + 41$ is a prime number.

$Q_n : (a + b)^n = a^n + b^n$

$R_n : 1 + 2 + 3 + \dots + n = \frac{n^2 + n - 6}{2}$

$S_n : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ (Fleming & Varberg 1989: 427)

The following type of questions were asked regarding these statements

Worksheet 9.3, question 1.

Let P_n represent the statement that $n^2 - n + 41$ is a prime number.

i.e. $P_n : n^2 - n + 41$ is a prime number.

1.1 Is P_1 true?

1.2 Is P_2 true?

1.3 Is P_3 true?

1.4 Is P_n true for all n ?

A statement may be regarded as a sentence which is either true or false. The principle of mathematical induction deals with a sequence of statements. In a sequence of statements, we have a statement corresponding to each positive number as we saw in the example above.

Visualising, patterning and generalisation are all utilised in Worksheet 9.4 to lead students to approach the principle of mathematical induction. In this way they are encouraged to engage in the third level activities of identifying hypotheses, intermediate deductions and conclusions.

The following examples of dominoes in Worksheet 9.4 illustrate P_n , Q_n , R_n and S_n of the previous worksheet.

Why They Fall and Why They Don't


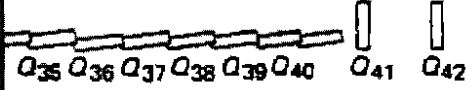


$P_n : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ <p>P_1 is true</p> <p>$P_k \Rightarrow P_{k+1}$</p>	 <p>First domino is pushed over. Each falling domino pushes over the next one.</p>
$Q_n : n^2 - n + 41 \text{ is prime.}$ <p>Q_1, Q_2, \dots, Q_{40} are true.</p> <p>$Q_k \Rightarrow Q_{k+1}$</p>	 <p>First 40 dominoes are pushed over. 41st domino remains standing.</p>
$R_n : (a+b)^n = a^n + b^n$ <p>R_1 is true.</p> <p>$R_k \Rightarrow R_{k+1}$</p>	 <p>First domino is pushed over but dominoes are spaced too far apart to push each other over.</p>
$S_n : 1 + 2 + 3 + \dots + n = \frac{n^2 + n - 6}{2}$ <p>S_1 is false.</p> <p>$S_k \Rightarrow S_{k+1}$</p>	 <p>Spacing is just right but no one can push over the first domino.</p>

figure 31 (Fleming & Varberg 1989: 429)

Further, a visual approach was included in order to encourage students to formulate arguments based on diagrams or visual representations. This was done last as it had become evident in the general case of block diagrams for geometric series that this was a challenging level three activity and not appreciated by all students. This approach may be found in Appendix A.

The students were given several opportunities of proving results by mathematical induction in Worksheet 9.6. Finally, in the last question of 9.6, the students were given the chance of passing through all the levels in the last question. First they had to study the diagrams and find the pattern and this led to establishing formulae for T_n and S_n . To complete the question

they had to perform the fourth level activity of proving the result by mathematical induction.

3. Consider the sequence represented below:

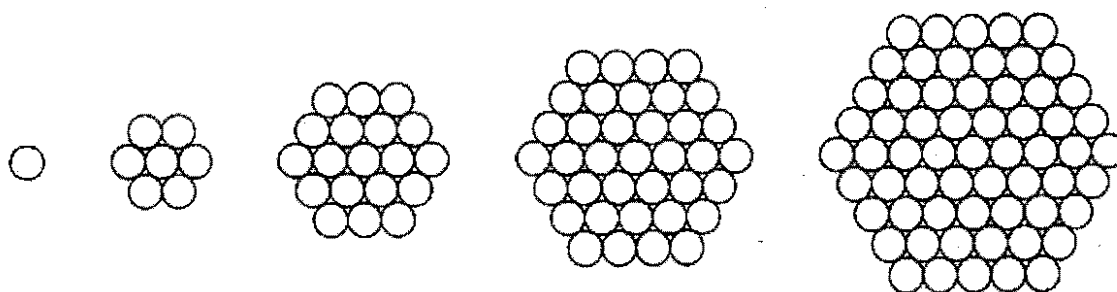


figure 32

(Jacobs 1994: 94)

- 3.1 Find the general term T_k for this sequence.
- 3.2 Find the formula for S_n for the terms of the series.
- 3.3 Prove your result in 3.2 by means of Mathematical Induction.

4.2.5 Conclusion

Students would have had to have developed a long way to be able to answer question 3 of Worksheet 9.6. Initially they would only have been able to detect a pattern in the number of circles. However, they would need to know about arithmetic sequences and series to find general formulae and to understand mathematical induction to prove the end result. Another reason why students would need to be on the fourth level to do this problem is they would have to relate different pieces of knowledge they had acquired to prove the truth of the formula for S_n . In Chapter 4 the effects of visualisation, patterning and generalisation on the advancement of students' mathematical thinking from one level to another in the topic of sequences and series will be researched.

4.3 Qualitative research

4.3.1 Introduction

Ever since the late 1970's, qualitative research has become an accepted tool used in educational research. This has come about because of the need to address those aspects of the human condition that require not just counting but also understanding. Since qualitative research attempts to appreciate human meaning, it is relevant for educational research

research attempts to appreciate human meaning, it is relevant for educational research regarding concepts with diverse meanings.

4.3.2 Justification for use of qualitative research

Qualitative research is distinguished from quantitative research in that quantitative research involves frequency whereas qualitative research is concerned with abstract characteristics of events. Qualitative researchers believe not only that many natural properties cannot be expressed in quantitative terms but that they would lose their reality if expressed merely in terms of frequency. As qualitative researchers direct their attention to the meanings given to events by participants, they reach a better understanding than they could have acquired through a list of descriptions or a table of statistics.

Quantitative researchers try to control the field of study to the greatest possible extent. They are detached and objective observers, restricting their attention to only one or more variables. However, this approach has been accused of being too narrow and not really objective because the mere presence of an observer changes the instruction being observed. The following criticism has been levelled at quantitative researchers: "Quantitative researchers are cast as number crunching, neo-know-nothing objectivists" (Cizech 1995: 26).

Whereas in the case of quantitative research, problems and hypothesis are always stated before the empirical investigation begins, in qualitative research these are flexible and may be reformulated during the inquiry. Criticisms have been levelled at qualitative research too. Berliner, commenting on the popularity of qualitative research writes: "We must learn to tell stories about our research" (Cizek 1995: 26). However, qualitative research is one of the many ways of gathering and interpreting information. Further, it might be said that quantitative and qualitative methodologists are often investigating the same things or different facets of the same phenomena.

4.3.3 Conclusion

Qualitative methods are used to understand human phenomena and investigate the meaning people give to their experiences. Piaget made use of qualitative research to test his theory of cognitive development. Although qualitative research may be subjective, it nevertheless can be reliable and valid. Consistency may be achieved by raw data being coded in such a way that others may both understand it and arrive at the same themes and conclusions. Here answers to questionnaires will be grouped, rated and depicted as bar graphs in order to give an indication of results of findings. Although the number of students is very small and the evaluation at times somewhat subjective, nevertheless the graphs will give some sort of indication of the success of the methods employed. Since the questionnaires are intensive and were provided at frequent, regular intervals, they will help to provide an assessment of overall student progress during the whole course of the lessons.

4.4 Research group

Prior to the series of lessons, a group of six students from a local secondary school were invited to participate in the study. The research group for this study was composed of six higher grade mathematics students from a co-educational high school in Centurion, South Africa. There were three girls and three boys and the students were of ranging abilities. For most of the lessons they were arranged into two groups of three students each. One group was composed of two girls and one boy while the other one consisted of one girl and two boys. The grouping of the students was done after two lessons, according to their abilities and personalities which had been reflected in the initial lessons. Students spent time having group discussions as they did their worksheets. The different letters of C, J, K, L, N and T are used to represent students instead of using their actual names. Fourteen one and a half hour lessons were held during the months of July and August, 2000. There were two lessons per week except during the last week of the school holiday and all students attended every lesson. The topics covered were sequences, series and mathematical induction. These were appropriate as the students had not as yet been taught sequences and series because it is usually a topic dealt with very hastily by the school right at the end of the year, just before the

final examinations.

4.5 Research instruments

Students were given seven questionnaires to answer at various stages of their lessons. These were designed to gauge their responses to the various topics covered. The van Hiele levels and progress through these levels were taken into consideration. Emphasis was placed on the three vital aspects of visualisation, exploring patterns and generalisation in the development of mathematical concepts and the advancement to higher levels of thought. The effects of these three important factors will each be dealt with separately in the next chapter.

4.6 Conclusion

The three main categories of research in Chapter 5 will include the effects of visualisation, exploring patterns and generalisation in the advancement of students through the pre-descriptive, descriptive, informal theoretical to the formal theoretical levels mentioned in this chapter. Questions related to advancement from each of the first three levels (levels 0, 1 and 2) to the subsequent ones (levels 1, 2 and 3) will be considered under each category. Ratings will be attached to rises from one level to the next and these results will be depicted by means of bar graphs. Conclusions will be drawn and recommendations made in Chapter 6. In addition, the effectiveness of different types of visual approaches will be considered.

CHAPTER 5

Data analysis

5.1 Introduction

In this chapter the student responses to questions in the questionnaires will be analysed. This will be done in order to attempt to measure the effect of emphasising visualisation, exploring patterns and generalisation on progress through the four levels 0 to 3 called the pre-descriptive, descriptive, theoretical informal and theoretical formal levels. These are the 4 levels which have been adapted from van Hiele's levels and utilised by Land. The objects and characteristics associated with each of the four levels in the topic of sequences and series have been listed in Chapter 4. The questionnaires were detailed and provided at frequent, regular intervals to effectively cover student progress. Although a sample group of six students is very small, it should nevertheless give some indication of the results of the study.

There are 7 questionnaires which are contained in the Appendix B. These are composed of questions which relate to the effect of visualisation, exploring patterns and generalisation on progress through levels. Ratings of 1 to 5 were given to questions asked in various categories. In many cases answers involve "yes" or "no" with an explanation. In these cases (in order to try to categorise answers for interpretation) the marks of the ratings will be awarded as follows.

1. "no" or "no" without a suitable explanation or no response at all.
2. "no" with a reasonable explanation.
3. "yes" or "yes" without a suitable explanation.
4. "yes" with a suitable explanation.
5. "yes" with a good explanation.

For example, when students were asked whether the study of patterns had helped them to see the need to prove results in mathematics, "1" was awarded to the answer "no", "2" to the answer "no, by studying a pattern you were not getting a result" and "3" for "yes". When

students were asked to explain the meaning of the *k*th, *n*th and *general terms* of a sequence, "4" was given for the answer. "The *k*th term is any term in the sequence which provides a general solution or answer for all the terms in the sequence" while "5" was awarded to the answer "The *k*th term and the *n*th term are the same thing, the letter *k* or *n* is used to represent any term in a sequence. The general term of a sequence gives the equation that can be used to determine any term in the sequence."

In other questions where not merely yes or no with an explanation were required, ratings were given according to the quality of the answer. Marks were deducted for any possible omissions made. Since these responses have no definite scale and would be determined by the writer, there could be some subjectivity involved. Nevertheless this was done to give a graphical representation of responses. For example, a rating of "1" was given for the response " $T_1, T_2, \dots, T_{19}, r, T_{21}, \dots$ " to the question, "Explain the difference between $T_{20} = r$ and $T_r = 20$ ". A "2" was awarded for the answer " $1 + 2 = 3, 1 + 3 = 4$ and so on" to the question about how the rows of Pascal's triangle were formed because the student indicated some idea of how it was done but gave an incomplete explanation. When students were asked whether $T_k = 2 \cdot 3^{k-1}$, $T_k = 6^k$ and $T_k = 5 \cdot 7^{-k}$ all were general terms for geometric sequences, "3" was given for the answer "1 : fits the formula - 2 : yes if it had been simplified - 3 : I'm not sure", "For the same question "4" was given for the answer "yes, they have powers" and "5" for "Yes, all can be made into the original formula". Occasionally students were asked questions not requiring any explanations. These sort of responses are at times included here but are not utilised in the ratings. Categorising of all answers appears in a table at the end of each subsection.

Questions from the questionnaires will be subdivided under headings of visualisation, exploring patterns and generalisation. Each of these categories will be subdivided into progress from levels 0 to 1, 1 to 2, 2 to 3 as well as general and in addition each of these cases will be accompanied by a table of ratings and a graph to illustrate the effectiveness of emphasising visualisation, exploring patterns or generalisation on progress from one level to

the next and through all the levels in general.

5.2 Visualisation

The effect of emphasising visualisation on the progress of students through levels of thought will be investigated here.

5.2.1 The progress from level 0 to level 1

After being provided with visual illustrations to help them establish the concept of a sequence in the first questionnaire, students were asked:

2. Did the use of visual representations (blocks, pictures, etc.) help you to understand the meaning of:

2.1 A sequence? Explain

2.2 The n th term of a sequence? Explain"

The students responded to these questions as follows:

2.1

- C: Yes, it allows you to recognise it more easily.
- J: Yes, it helped me discover the pattern.
- K: Yes, it helped you to work out the pattern.
- L: Yes, it makes the sequence more recognisable.
- N: Yes, it shows the ordered repetition / list visually.
- T: Yes, it helped me to see the pattern.

2.2

- C: Yes, patterns in the sequence can be seen more easily and the n th term worked out.
- J: Yes, we could deduce a pattern and thus the resulting sequence.
- K: Yes, from the picture we could deduce the pattern and thus the resulting sequence.
- L: Yes, visually recognisable patterns can be seen e.g. 3 above and 2 across, adding or multiplying, etc.
- N: Yes, the first terms established the pattern but when working with the n th term the numbers became too great and you establish the n th term by using the numbers that follow.
- T: Yes, from the pictures you can deduce the pattern and resulting sequence.

Students responded positively and thus it appears that students found the use of visual examples helpful in progressing from level 0 to level 1. They were encouraged to think about relevant aspects of visual illustrations. In the second questionnaire the following question was asked:

4.1 What aspect(s) of the visual representation above help(s) us to identify and continue the sequence?

4.2 How would we arrive at a formula for the n th general term of this sequence?

The following student responses were recorded:

4.1

- C: *By counting how many circles the picture before increased, you add an extra ring of circles around the previous one.*
- J: *The way the number of circles on the sides of the hexagons increase per term.*
- K: *You can see that each time you add another round of circles.*
- L: *The shape and counting the no. of times it increases in T_1, T_2 & T_3 .*
- N: *The growing size of the objects. We find the first few terms and calculate the rest in our minds.*
- T: *The arrangement and increasing number of balls.*

4.2

- C: *You count all the circles in each picture then work out the difference of each picture. Each one increases by a multiple of 6.*
- J: *We were counting the circles on the sides of the hexagons and relating them to the term number.*
- K: *You count the number of circles that you add on each time and see how they relate to each term previously.*
- L: *Count the number of circles. Then determine the difference between the numbers.*
- N: *We first found the pattern, then calc.*
- T: *By counting the amount that each term increases by.*

At the end of the final lesson, all students responded "yes" to the use of visualisation to establish rules as indicated by question 6.2 of Questionnaire 6:

Did the use of visual illustrations help you to see the need to establish rules?

However, a few weeks later the below responses to question 4.2 of Questionnaire 7 seem to indicate that although they had previously found visualisation effective in helping them understand, identify and continue sequences as well as develop formulas for their n th terms, it did not necessarily show them the need to establish rules.

"Did the provision of visual illustrations help you to see the need to establish rules? Explain.

4.2

- C: *Yes, you could see the progression.*
- J: *No.*
- K: *Yes, because by copying the rule you can see how the illustration works.*
- L: *No*
- N: *Do not really, making a mind picture is only making the problem easier and I don't think it helped me establish a rule.*
- T: *No.*

N's response suggests that visualisation might make the topic appear easier but it may not

necessarily help establish a formula as this would require more abstract thinking involving generalisation. The following table summarises the student response ratings with the corresponding graph below it. The ratings of responses for each individual student appear in the table. Each "1", "2", "3", "4" and "5" was counted and the results were indicated at the bottom of the table. The graph seems to suggest that with the exception of some negative responses to the use of visualisation for showing the need to establish formulae, students felt that the promotion of visualisation had made a significant contribution to their rise from the pre-descriptive to the descriptive level.

Visualisation : the progress from level 0 to level 1							
Questionnaire	number	C	J	K	L	N	T
1	2.1	5	5	5	5	5	5
1	2.2	4	4	4	4	5	4
2	4.1	5	5	5	5	5	5
2	4.2	4	5	5	4	3	4
7	4.2	5	1	4	1	2	1

Ratings	1	2	3	4	5
number of responses	3	1	1	9	16

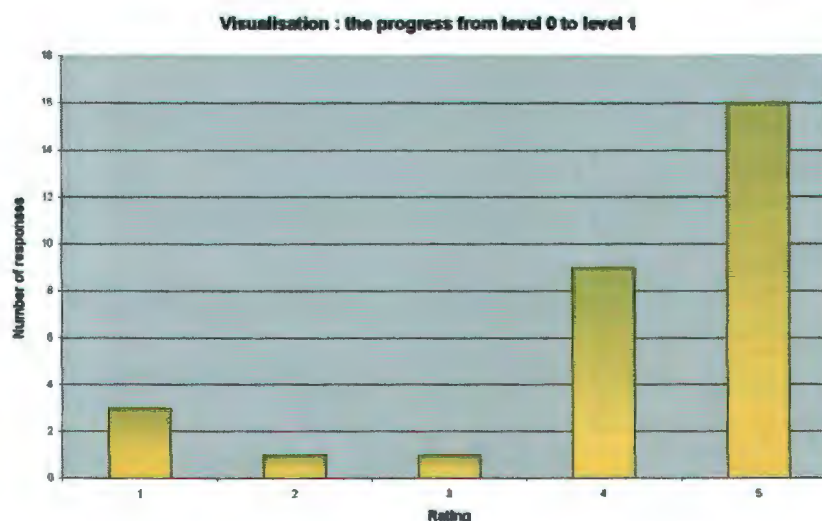


figure 33

5.2.2 The progress from level 1 to level 2

Pupils had thus begun to see the use of visual examples as being helpful in their understanding of the concept of a sequence and the meaning of a formula for the n th term of a sequence. However, they could not all see how this would assist them in doing examples in which the rules were utilised. Below are some of the responses to question 11.1 of Questionnaire 2.

"Did visualisation help you to do the examples in worksheets 2.4 and 3.4? Explain."

- C: Yes, by working out the difference you can see how the sequence is going to develop.
 J: No, due to the fact that visualisation was totally absent,
 K: No, there was no visualisation given.
 L: Yes, by picturing what is actually happening, it is easier to find a solution.
 N: Yes
 T: No

Subsequently, a discussion was held on what the meaning of visualisation actually is. In the next questionnaire 3 the following responses were recorded to the questions 1.1 and 7 of Questionnaire 3.

1.1 What is visualisation? and

7. Did the use of visualisation help you to do the worksheets on geometric sequences? Explain."

1.1

- C: Seeing / picturing something in your mind's eye.
 J: Something you can see which is represented visually.
 K: Something that you can picture in your mind.
 L: When you picture something in your head to try and understand it better.
 N: Is the mind's picture / means of starting, a basis for you to start from.
 T: A mental picture in your head.

7

- C: Yes, because you could see what the difference / common ratio was.
 J: No.
 K: Yes, you could picture where the term is in the sequence.
 L: No, you just used the numbers on the page.
 N: Yes, it helped with the first terms.
 T:

The students were thus of divided opinions as to whether visualisation was of relevance when working on applications at levels one and two. Some believed it helped them to try and envisage what a sequence would look like in their mind's eye but others still believed that visualisation did not serve a purpose when working on a more abstract level with symbols.

Students responded with great enthusiasm to the visual approach to establishing the summation formulae for arithmetic and geometric series. The following answers were recorded in response to question 2 of the fourth questionnaire:

2. "Did the visual approach help you to appreciate the summation formulae for:
 2.1 arithmetic series? Explain.

2.2 geometric series? Explain."

2.1

- C: Yes, because you can visually see how many blocks there were then apply it.
- J: Yes! it helps us understand and develop our own formulas.
- K: Yes, it helps you to see what you have to do to get the formula.
- L: Yes, it helped to derive the formulae, while you can see what is happening.
- N: Yes, using the squares established a good base to work from.
- T: Yes, it helps you to see the formula.

2.2

- C: Yes, you could see that from each one you had to subtract 1.
- J: Yes! it helps us understand and develop formulae and visualise certain values thus enabling us to predict future terms.
- K: Yes, same as above.
- L: Yes, you can see why you say something (i.e. in algebraic terms).
- N: Yes, using the squares established a good base to work from.
- T: Yes, same as above.

The students generally found that visual illustrations provided them with a good base and gave them insight regarding the establishment and meaning of formulae at level two. They were able to define relevant terms as well as appreciate the derivation and application of the summation formulae for arithmetic and geometric series. This seemed to indicate a successful rise from the descriptive to the theoretical informal level. The table and graph that follow give an indication of the student responses. The few ones and twos show that some students were not certain as to whether visualisation would help them with ordinary applications of formulae. However, the many fours and fives give an indication of the success of the visual introductions of the arithmetic and geometric series formulae. Although the block diagram for the n th term of a geometric series was found rather difficult to interpret, students formed the abstraction based on the numerical values they had deduced from the results from the first few diagrams rather than trying to interpret the result from the diagram reflecting the n th case. Forming an argument from such a visual representation definitely would seem to involve theoretical formal level thinking of a nature quite complex and foreign to students. The results below suggest that visualisation does seem to assist in rising from level 1 to level 2.

Visualisation : the progress from level 1 to level 2							
Questionnaire	number	C	J	K	L	N	T
2	11.1	5	2	2	5	3	1
3	1.1	5	5	5	5	5	5
3	7	4	1	5	2	4	1
4	2.1	5	5	5	5	5	4
4	2.2	5	5	4	5	5	4
Ratings		1	2	3	4	5	
number of responses		3	3	1	5	18	

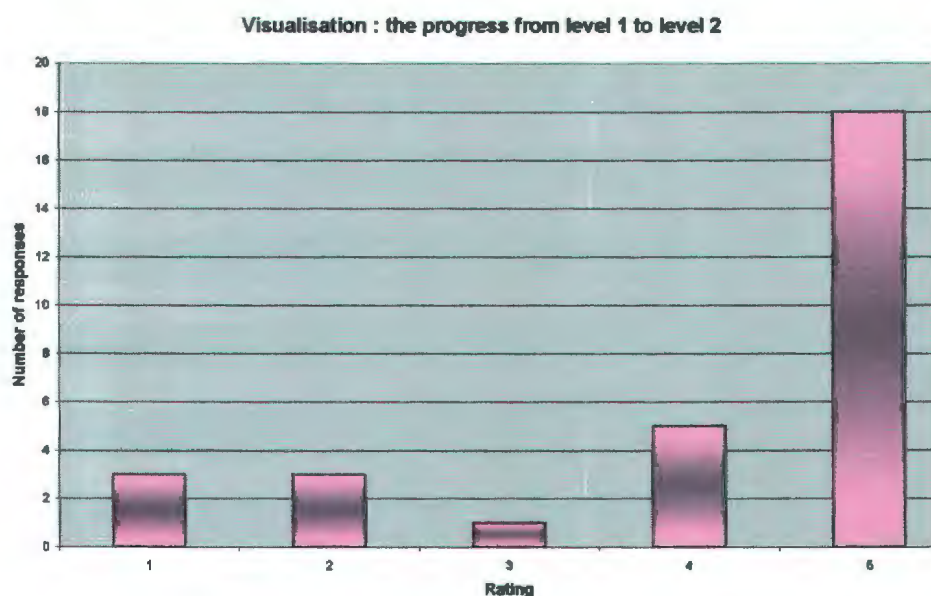


figure 34

5.2.3 The progress from level 2 to level 3

In order to rise to level 3, various illustrations were used to lead to the concept of a limit.

These included: many numerical values sometimes using a calculator but primarily using a computer; block illustrations and graphs. The pupils' responses to the computer values chosen are recorded in question 3.2.1 of the fifth questionnaire:

3.2.1 Did the below computer values help you to visualise that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1? \quad \text{Explain.}$$

- C: Yes, you can see it is heading towards 1 - a fixed value.
 J: Yes! we can see that it heads towards 1 but never reaches it.
 K: You can slowly see the values heading towards 1.
 L: Yes, you can see the number getting closer and closer to 1.
 N: Yes, the limit is 1.
 T: Yes, you can clearly see that the sequence tends towards 1.

There were mixed feelings regarding the use of block diagrams to illustrate limits as may be seen in the following responses to following question 3.2.2 of Questionnaire 5.

3.2.2 Did the blocks represented below help you to visualise that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1? \quad \text{Explain.}$$

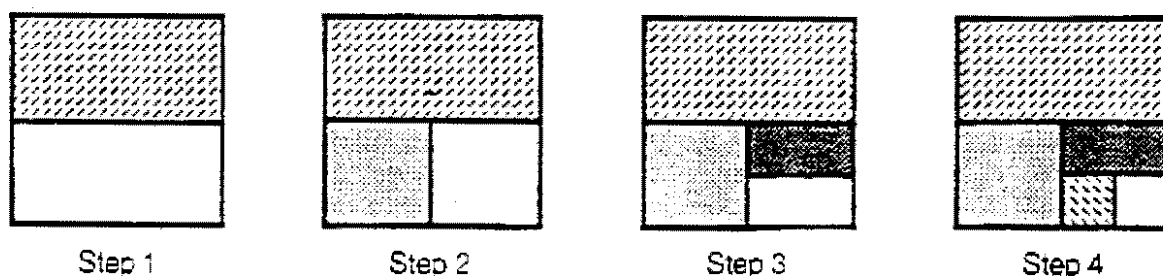


figure 35 (Bennett 1989: 134)

- C: Yes, each time the shaded block became less and tends towards 1.
 J: No! They were confusing.
 K: Yes, slowly see all the blocks being shaded.
 L: No.
 N: Yes, the steps tend towards a whole 1, but it should never reach 1.
 T: No, didn't understand.

The students did appreciate the graphical illustrations as is indicated from their answers to question 3.2.3 of Questionnaire 5.

3.2.3 Did the graphical method help you to visualise that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1? \quad S_n \text{ is plotted against } n :$$

- C: Yes, the graph curves towards 1.
 J: Yes, the graph allows us to see that the limit heads towards 1.
 K: Yes, the graph begins to curve at 1.
 L: Yes, you can see that the graphs evens out.
 N: Yes, the values limit is 1, it will never ever reach it.
 T: Yes, you can see that the slope heads towards 1.

K and T appeared to understand the concept but were unable to express their thoughts in a correct mathematical way. When students were asked whether or not they were in favour of several means of visual representation, they were all convinced that it was helpful and beneficial to them as illustrated by their responses to question 3.4 of Questionnaire 5.

3.4 Do you think it is helpful to be provided with various means of visual illustrations such as in example 3.2 above?

- C: Yes, it helps to understand what is going on better.*
- J: Yes, as some people interpret certain things differently.*
- K: Yes, because each time it becomes more clear.*
- L: Yes, you get a better idea.*
- N: Yes, it gives us a better way of understanding by repetition of what is the same formula.*
- T: Yes*

Visual illustrations were also provided to lead pupils to the third level activity of proof by mathematical induction. The domino illustration was positively accepted by students as seen in their answers to question 4.3.1. of Questionnaire 6.

4.3.1 Did the domino illustration help you to understand proof by induction? Explain.

- C: Yes, you could visualise what had to happen for it to be true.*
- J: Yes! We could visualise that if it hits down the next domino then it is true.*
- K: Yes, you could visualise what was happening and see if it was going to be true.*
- L: Yes, it helps you picture why it might not work for some values and it does for others.*
- N: Yes, it showed there might be a breakdown somewhere in the chain.*
- T: Yes, when you see a practical explanation it helps.*

Once again, the block illustration was not as popular with students as illustrated by their responses to question 4.3.2 of Questionnaire 6:

4.3.2 Did the block illustration help you to understand proof by mathematical induction? Explain.

- C: Yes, it is logical.*
- J: No! Because I didn't get it.*
- K: Yes, it is logical.*
- L: No, it was too complicated.*
- N: No, it confused me.*
- T: Yes, you can see the pattern.*

Below are the ratings and graph connected to the effect of stressing visualisation in rising from level 2 to 3. Despite the students' inability at times to see the connections between visualisation and the more abstract activity of proving, visualisation seems to have been effective in promoting the rise from the theoretical informal to the theoretical formal level.

Visualisation : the progress from level 2 to level 3							
Questionnaire	number	C	J	K	L	N	T
5	3.2.1	5	5	5	5	5	5
5	3.2.2	5	1	5	1	5	1
5	3.2.3	5	5	4	4	5	4
5	3.4	4	4	5	4	5	3
6	4.3.1	5	5	5	5	5	5
6	4.3.2	5	1	5	1	1	5
Ratings		1	2	3	4	5	
number of responses		6	0	1	6	23	

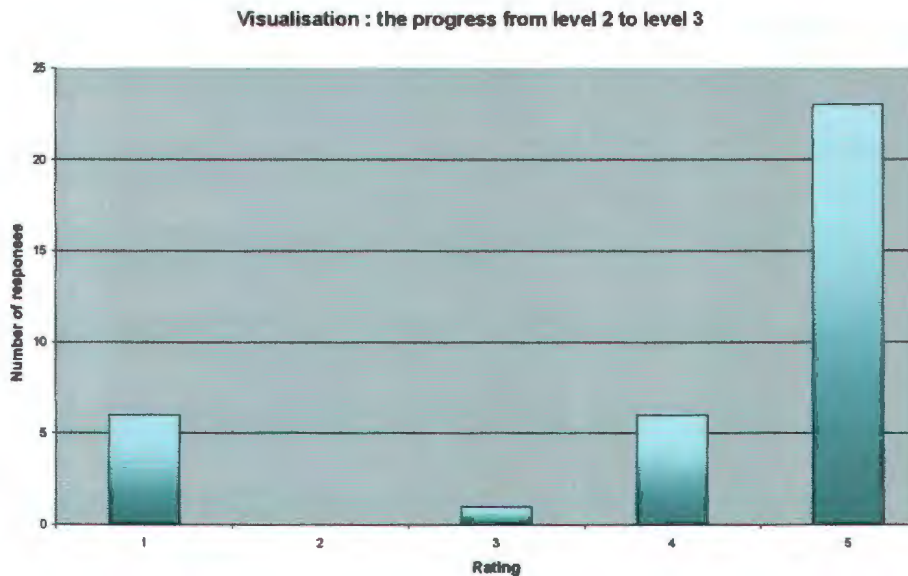


figure 36

5.2.4 The effectiveness of different types of visualisation.

The various forms of visual representations were examined to determine what appealed more strongly to the students. The concept of a limit of a sequence was introduced by using many numerical values sometimes using a calculator but primarily using computers. For example, the students responses to question 2.2 of Questionnaire 5 below indicate that they had found the computer values to be very helpful in this regard.

2.2 Did the computer values for this and other sequences help you to appreciate the idea of a limit?

All students agreed that the computer values had helped them to appreciate that the limit was the golden ratio. L's response follows below:

2.2

L: Yes, it demonstrated how a sequence can get closer and closer to a number.

Unfortunately these responses could not be used in the ratings because too few

explanations were given.

Infinite sums of series were illustrated by computer values, block diagrams and graphical representations and the student responses to these have already been recorded in 5.2.3.

The students were questioned about which they regarded as being the most effective visual representation of the concept of a limit. Their responses to question 3.3 of Questionnaire 5 are recorded below:

3.3 Which of the methods in 3.2 i.e. 3.2.1; 3.2.2; 3.2.3 did you find the most effective in

illustrating that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$? Explain.

C: The computer because you can see it down to the last decimal, it is also more accurate.

J: 3.2.1 - one can easily see the values converging towards 1.

K: The computer because it is with numbers and you see it from one decimal to the next.

L: Computer - you can see definite values.

N: The graph it physically / visually shows how the limit is converged upon.

T: 3.2.1 and 3.2.3. - They are most easy to understand and see.

The above answers (which were not possible to include in the ratings) show that the computer definitely provided the most appreciated visual illustration and the graphical method was the second most popular one.

Interpreting limits using all three of the above methods involves the third level activity of formulating an argument based on diagrams or visual representations. Formulating such an argument based on a large quantity of numbers approaching a specific value or seeing a graph heading in a specific direction (to a slightly lesser extent) seems to be a more natural thing for most students to do. However, visualising blocks behaving in a certain way seems to be a more difficult activity for certain students. This could possibly be because they are unused to making deductions of this kind or because making deductions from a block diagram, particularly when it represents the n th case, is a more difficult abstraction to make as it seems to leave more to the imagination. This at times led to problems forming arguments based on such diagrammatic representations. Consequently, as time progressed, not too many of these were utilised.

When mathematical induction was introduced both the domino and the block illustration were used and the students' interpretation of these results were recorded in 5.2.3. All but one of them felt that the domino illustration had given them a better understanding of proof by mathematical induction but T claimed he understood the block illustration as well. As indicated in 5.2.3, students were in favour of being provided with several types of visual representations to formulate concepts. They felt that it catered for the different ways in which people see things, increased their understanding and each time made the concept become clearer.

5.2.5 General

The responses of students to more general types of questions are recorded below. These are concerned with their perception of the overall worth in helping them establish concepts and develop insight into topics. Every student gave a definite positive "Yes" response to questions 6.1 of Questionnaire 6:

6. Did the provision of visual illustrations help you to
6.1 understand the concepts we have learnt?

Unfortunately these results could not be counted in the ratings because no explanations for the answers were requested. However, when the questions were repeated in Questionnaire 7 and an explanation required, their responses were once again positive as indicated below.

- 4.1 Did the provision of visual illustrations help you to understand the concepts we have learnt? Explain.

4.1

- C: Yes, because you could actually see what was happening
- J: Yes! the brain relies on visual illustrations.
- K: Yes, when you see something usually it become easier to understand.
- L: Yes, can see what you are working with.
- N: Yes, partly, I think the blocks and circles which grew or decreased helped a lot.
- T: Yes, recognising patterns.

Students acknowledged the benefit of their visual experiences as is evident in their responses to final question 9 in Questionnaire 7:

9. Did the use of visual illustrations give you insight into the topic of sequences and series?

- C: Yes you could in your mind's eye see how the pattern would progress.
- J: Yes. One can visualise a sequence increasing.

- K: Yes, by first understanding the visual illustrations we could get a better knowledge and insight into sequences and series.*
- L: No, if they form patterns or represent an idea that we are learning, then Yes.*
- N: Yes, it gives you a basis to go by and start solving the problem.*
- T: Yes, it's easier to identify a sequence by looking at a visual pattern.*

The responses of the students to the questionnaires definitely revealed that emphasis on visualisation had helped them move from one level to another. It helped them to appreciate rules, derive formulae, understand the concept of limits and proof by mathematical induction.

The below general graph, showing the overall effect in rises from each level to the next seems to indicate that visualisation had a very positive effect on the overall development of concepts. This graph includes all the ratings from levels 0 to 1, 1 to 2 and 2 to 3 as well as the general questions (questions 4.1 and 9 of Questionnaire 7).

Visualisation : General		C	J	K	L	N	T
Questionnaire	number						
7	4.1	5	5	5	5	4	4
7	9	4	4	5	3	5	4
Overall Ratings		1	2	3	4	5	
number of responses		12	4	4	25	63	

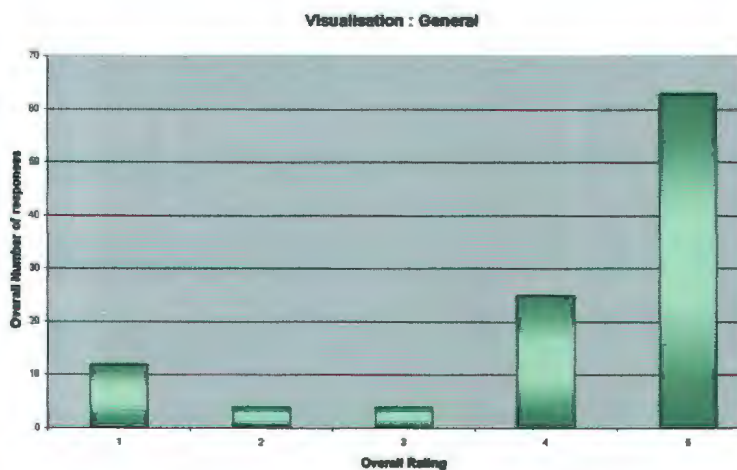


figure 37

5.3 Exploring patterns

The effect of exploring patterns in encouraging pupils to move from one level to another will be studied here. In the lessons of Chapter 4, numerous patterns were provided for this purpose.

5.3.1 The progress from level 0 to level 1

In the first questionnaire students were questioned to determine whether they had developed an understanding of the concepts of a sequence and the general term of a sequence.

3. Did the study of patterns help you to understand the meaning of numbers that follow:

3.1 A sequence? Explain.

3.2 The n th term of a sequence? Explain.

3.1

C: Yes, it allows you to see what is happening in the sequence.

J: Yes! we could deduce a pattern and thus the resulting sequence.

K: Helped you work out the pattern

L: Yes. Shows symmetry or other patterns and orders of objects.

N: Yes, it shows how objects can be placed symmetrically etc. and it shows the order used in a sequence.

T: Yes, it help me see the pattern.

3.2

C: by recognising the pattern to a formula can be determined and the n th term calculated.

J: Yes! we could deduce a pattern and thus the resulting sequence..

K: Yes, from the picture we could deduce the pattern and thus the resulting sequence.

L: Yes, by recognising a pattern, a type of formula can be determined and thus the n th term calculated.

N: The pattern helped in establishing the first terms and the understanding to be able to have foresight and create a formula for the n th term.

T: Yes, from the pictures you can deduce the pattern and resulting sequence.

Students were presented with many different patterns, including some very challenging ones or ones for which they were not as yet able to find the general term. This resulted in some negative responses to question 3 of Questionnaire 2.

Questionnaire 2 question 3.

Are patterns always immediately obvious?

C: No, some are very complex and you need to first think about them or they require calculation.

J: No, some require careful scrutiny

K: No.

L: No, sometimes you have to search for a pattern as it might be quite complex.

N: No, some are more in depth and require calculations.

T: No.

The responses of students to question 3.2 once the series of lessons had all been completed

seem to indicate that the majority of students had found that the study of patterns had helped them appreciate the need to establish rules for sequences:

Questionnaire 7 no. 3.2.

3.2

C: Yes, because then you would have to carry on forever with the pattern.

J: No, not really.

K: Yes, when you first look at the pattern the rule becomes clearer.

L: Yes, you can see regular or repetitive ideas.

N: Yes patterns can be difficult to see at first but if you have a rule you can follow by solving any problem.

T: I can't see how.

Students had acquired an appreciation for the concept of a sequence and were able to understand and perform simple calculations with general rules. The graph of their ratings for the above questions follows below. There is a notable difference between this and the previous graphs. The ones and twos are due to the more difficult patterns involving, for example, arithmetic summation formulae for general terms. Although students were not required to find general terms in these cases, they became aware that sometimes searching for patterns could be challenging and finding the general term could be a fairly complex task. Despite this, from the graph it would appear that exploring patterns had made a valuable contribution to students' rise from level 0 to 1.

Exploring Patterns : the progress from level 0 to level 1							
Questionnaire	number	C	J	K	L	N	T
1	3.1	5	4	5	5	5	4
1	3.2	5	4	4	5	5	4
2	3.3	2	2	1	2	2	1
7	3.2	5	1	5	5	5	1
Ratings		1	2	3	4	5	
number of responses		4	4		5	11	

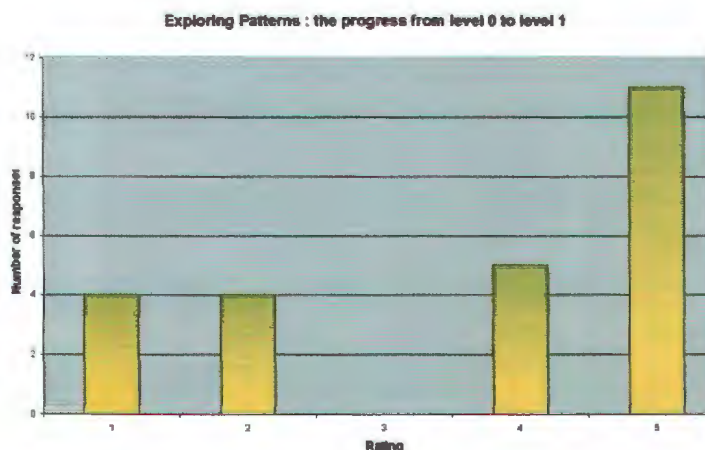


figure 38

5.3.2 The progress from level 1 to level 2

At level 2 students are supposed to be able to define relevant terms. After completing the initial worksheets on patterns and discussing the idea of a pattern amongst themselves, students answered the following question 1.1 of Questionnaire 1:

1. How would you describe a pattern?

- C: *An order that is repeated.*
- J: *A pattern is necessary to produce a sequence.*
- K: *A pattern is necessary to produce a sequence.*
- L: *An ordered list that is repeated.*
- N: *An ordered repetition of objects etc.*
- T: *The building blocks for a sequence.*

Thus the students were able to extract the essence of what a pattern was and gave a definition for it. In order to rise up from level 1 to level 2, they also compared different patterns. They were encouraged to analyse patterns and gave the following responses to question 1 of Questionnaire 2.

1. What properties do all patterns have in common?

1

- C: *They repeat each other, have a certain order.*
- J: *They all follow a formula which can help predict the next pattern in the sequence.*
- K: *The rest of the pattern which is not given can be worked out.*
- L: *There is always a repetition which makes a pattern visible.*
- N: *All patterns have something or other repeated.*
- T: *An order, or sequence, some are predictable.*

The students were questioned to see whether they felt that exploring patterns had helped them to operate efficiently on level 1 and rise to level 2. They gave the following responses to question 11.2 of Questionnaire 2:

11.2 Did exploring patterns help you to do the examples on worksheets 2.5 and 3.4?

Explain.

- C: *Yes it helps to recognise the patterns in the sequence.*
- J: *Yes! We have become used to looking for patterns and detecting sequences by exploring patterns.*
- K: *Yes, we could look for a pattern to help us carry on with the sequence.*
- L: *Yes, it helps you to be more aware of the types of patterns that can occur and helps you recognise other patterns and how they occur.*
- N:
- T: *Yes, it taught us to recognise patterns and ultimately sequences.*

The above answers seem to suggest that the pupils had become accustomed to looking for

patterns and this had helped them to do the more difficult level 1 and 2 examples contained in the worksheets. This idea was confirmed by some but not all of the answers to question 8 of Questionnaire 3:

8. Did exploring patterns help you to do the worksheets on geometric sequences?

Explain.

- C: *Yes, it helped to recognise that it was a pattern.*
- J: *No.*
- K: *No.*
- L: *Yes, easy to recognise.*
- N: *Yes, they helped to establish your visualisation.*
- T: *No.*

These responses suggest that some students did not believe that studying patterns had really helped them to do level 2 type problems on their worksheets as they were of a more abstract nature. Further patterns, such as the one found in Pascal's triangle, were studied to make students aware of different types of number patterns and prepare them for proof by mathematical induction. All students were able to complete the rows of the triangles correctly. Below are recorded their responses to question 1.3 of questionnaire 6:

1.3 Do you think the knowledge of Pascal's triangle is useful in mathematics? Explain.

All students were able to complete the rows of the triangles correctly.

- C: *Yes, it helped us to multiply out cubic, quad etc.*
- J: *Yes, it enables us to multiply out cubes and quads, etc.*
- K: *No.*
- L: *Yes, it helps you multiplying the terms and easily especially when you get above power 2.*
- N: *It helps to multiply out easier.*
- T: *Yes, helps you to determine the coefficients without multiplying out*

Hence students believed that the pattern of Pascal's triangle had great relevance in mathematical applications. This seems to suggest that patterning had indeed contributed to their rise to level two and the performance of level 2 based activities. Below is a table of ratings of questions relating to the rise from level 1 to 2. Here it would seem that exploring patterns had a very positive effect on the rise from levels 1 to 2.

Exploring Patterns : the progress from level 1 to level 2							
Questionnaire number		C	J	K	L	N	T
1	1.1	5	4	4	4	5	4
2	1	5	5	5	5	5	5
2	11.2	5	5	5	5	1	5
3	8	4	1	1	5	5	1
6	1.3	5	5	5	5	4	5
Ratings		1	2	3	4	5	
number of responses		4			6	20	

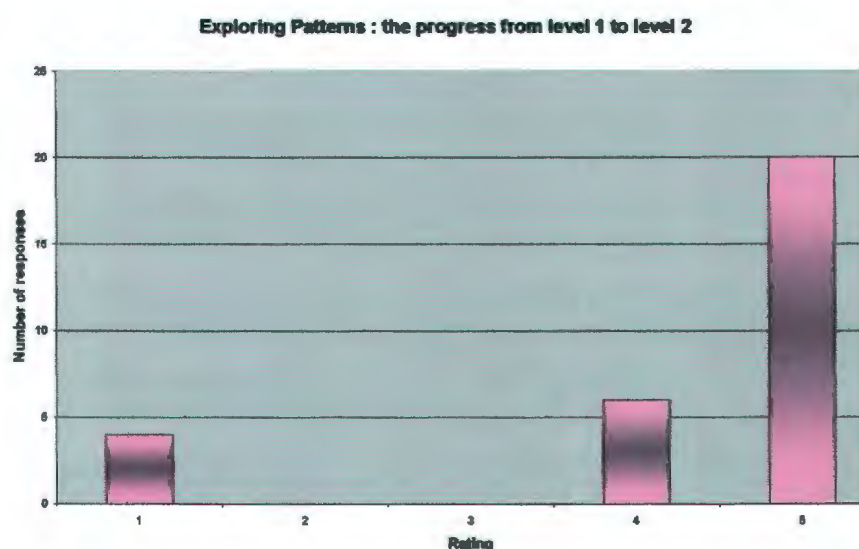


figure 39

5.3.3 The progress from level 2 to level 3

In the various topics that were taught, different activities were organised to promote advancement from level 2 to level 3. Being able to make arguments based on visual representation is a level three type activity and in questionnaire 6 the students were asked to:

1.2 Describe how Pascal's triangle is formed.

Their responses follow below.

- C: You always start with one. The term below = to the sum of the 2 above numbers. If there is no number we assume it is 0.
- J: The term below = the sum of the two above numbers. Where no no. is present, we assume that it is zero.
- K: The term below is equal to the sum of the two above numbers, we assume that were there no number that the number is zero..
- L: .By adding the two terms next to each other in the line above. Always start with a 1.
- N: .The above 2 are added together.
- T: $1 + 2 = 3$, $1 + 3 = 4$ and so on.

They all had the general idea but were not all able to give complete verbal descriptions of the pattern. The students certainly did appreciate that the pattern of the layout of terms in

Pascal's triangle makes it easier to continue the rows than the way suggested in question 2.1 of Questionnaire 6. The responses to question 2.2 are recorded below:

2.2 Which of the ways 1.1 & 1.2 above depicting Pascal's triangle is more effective?

(Q1.1 Or Q1.2) Explain.

C: *The apex is the centre of the base*

J: *Writing it with the apex in the centre of the base.*

K: *Writing with the apex in the centre of the base*

L: *1.1 You can see exactly what's going on.*

N: *1.1 The addition of the above terms is much easier.*

T: *1.1 You can easily see how to get the other numbers by adding the above two.*

Thus the students were convinced that patterning played a major role in the establishment of Pascal's triangle.

After all the lessons had been completed, the students were asked the following question in Questionnaire 7:

3. Did the study of patterns help you to

3.3 see the need to prove results in mathematics? Explain.

Their responses were as follows:

C: *Yes, because otherwise we would not have known if the answers were right.*

J: *No, I couldn't see the connection.*

K: *No, by studying a pattern you were not getting a result.*

L: *Yes*

N: *No*

T: *I can't see how.*

Thus it seems that in retrospect, although most students claimed that exploring patterns had helped them understand relevant concepts and establish rules, they did not see a connection between exploring patterns and the need to prove results in mathematics. In other questions of their questionnaires they did express an appreciation for proof and the need to prove results but they seemed to regard proof as being more closely related to generalisation and more abstract thinking.

Exploring Patterns : the progress from level 2 to level 3							
Questionnaire number		C	J	K	L	N	T
6	1.2	4	4	4	4	3	2
6	2.2	4	4	4	4	5	5
7	3.3	4	1	2	3	1	1
Ratings		1	2	3	4	5	
number of responses		3	2	2	9	2	

Exploring Patterns : the progress from level 2 to level 3

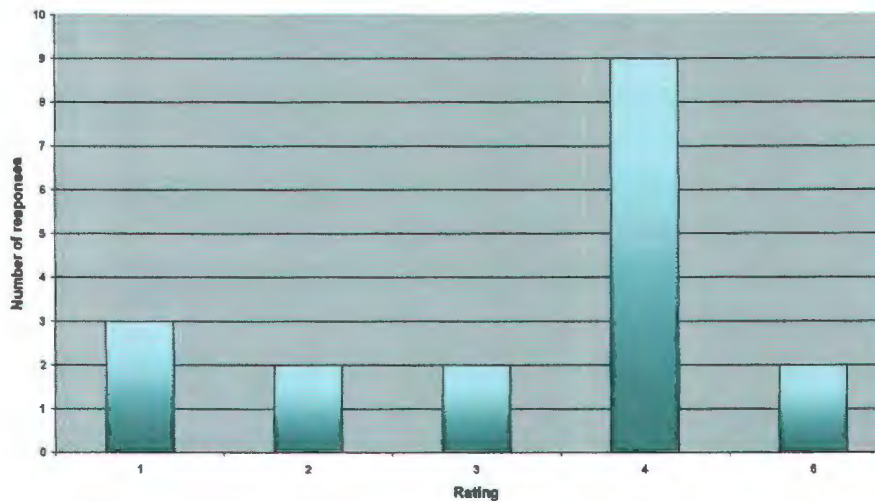


figure 40

5.3.4 General

After the lessons had been completed, one of the general questions on the last Questionnaire 7 was as follows:

3. Did the study of patterning help you

3.1 understand the concepts we have learnt? Explain.

The student responses to this question are recorded below:

C: Yes, because we could understand what was going on better.

J: No, Just because

K: Yes, it helped us to understand the basics.

L: Yes, it made it easier to see.

N: Yes, they give a good basic understanding.

T: Yes, learnt to recognise patterns.

The majority of pupils seemed to feel that in retrospect the exploring of patterns had helped to form a good basis for their understanding of the work covered in the lessons. The below ratings and graph cover all questions asked in levels 0 to 1, 1 to 2 and 2 to 3 as well as the general question 3.1 of Questionnaire 7. The overall impression of the general graph certainly does suggest that exploring patterns did assist in the progress through the various

thought levels.

Exploring Patterns : General		C	J	K	L	N	T
Questionnaire number							
7	3.1	5	1	5	5	5	5
7	8	5	3	5	5	3	4
Overall							
Ratings		1	2	3	4	5	
number of responses		12	6	4	21	41	

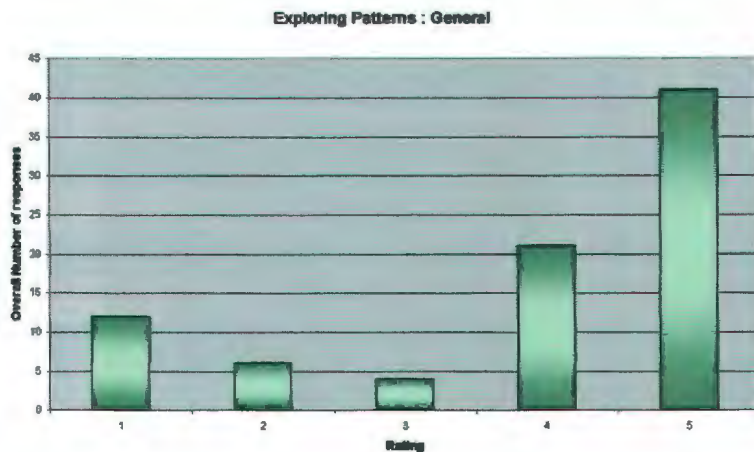


figure 41

Comparing all the graphs of progress in thought levels from 0 to 1, 1 to 2, 2 to 3 and general, it becomes evident that exploring patterns had been effective in moving through all the thought levels particularly levels 1 to 2. More difficult patterns and finding more complex rules caused some low ratings in progress from levels 0 to 1 while not quite perfect arguments with regard to visual representations and regarding patterns as not being directly linked to the need to prove results resulted in a decrease in the ratings of the graphs for levels 0 to 1 and 2 to 3 respectively. However, the graph reflecting levels 1 to 2 and the general graph indicate that on the whole exploring patterns had made a positive contribution to the movement through the different thought levels.

5.4 Generalisation

Generalisation was something which was being continually emphasised throughout the lessons in Chapter 4. It was stressed in order to encourage the progress of pupils from one thinking level to another. In the questionnaires students were asked questions designed to give an indication of their perception of generalisation at different stages of their learning.

5.4.1 The progress from level 0 to level 1

In order to facilitate the movement of students from level 0 to level 1, they were encouraged to develop an understanding of a sequence, recognise different types of sequences and establish formulae for general terms for the sequences they studied. The following responses were given to the sixth question of Questionnaire 1:

6. Do you see any use for the n th term of a sequence? Why?

- C: Yes, it allows you to work out any term number without having to work all the terms out.*
- J: Yes because of the use of architecture and statistics to forecast.*
- K: Yes because of the use of architecture and statistics to forecast.*
- L: Yes. When you know the formula for the n th term, then any term can be calculated.*
- N: It is an easy way of processing numbers without having to go through the whole series of numbers.*
- T: Yes, statistics.*

The above responses show a real appreciation for the significance of the n th term of a sequence. Pupils initially established rules for sequences in general before they began to study specific types of sequences such as arithmetic and geometric ones. For example, they established the rule that $F_k = F_{k-2} + F_{k-1}$ for Fibonacci sequences but felt it was not of much help to find a term such as the 100th one. They gave the following answers to Questionnaire 2, questions 5.1 and 5.2:

5.1 What rule did we establish for the k th term of a Fibonacci sequence?

- C: The first 2 terms are given we then add the previous term to set the next one. e.g. 1; 1; 2; 3; 5*
- J: We add the previous terms.*
- K: That each time you add the 2 previous terms.*
- L: T_1 and T_2 are given, then you add the previous 2 terms i.e. T_{k-1} and T_{k-2} .*
- N: The first two terms are given, then as you get to the third term you add the previous 2 terms..*
- T: That each time you would add the previous two.*

5.2 Did it help us predict the 100th term easily? Explain.

- C: Yes because by using the formula $T_{n-1} + T_{n-2}$ you can substitute in.*
- J: No, we would have to know what the 98th and 99th terms were..*
- K: No, you would have to know the 98th and 99th term.*
- L: No, you have to know the previous two terms first..*
- N: No, we didn't have the 2 previous values..*
- T: No, you don't know the two previous two terms.*

Here the students revealed an understanding of the general term of a sequence and its potential power. In question 6 they were required to answer further more general questions regarding sequences:

6. Explain the meaning of the k th term, the n th term and the general term of a sequence.

- C: The formula that can be used to work out the value of any term.*
- J: The k th term and the n th term are the values of the term corresponding to the term number in the*

sequence. The general term is an equation involving the number of the term which can be substituted to work out the value of the term we are looking for.

- K: T_k term is the position it occupies in the sequence; general formula helps you find the answer to the formula you are working out. k th is the answer to the term (T_k).
- L: The k th term and the n th term are the same thing, the letter k or n is used to represent any term in the sequence. The general term of a sequence gives the equation that can be used to determine any term in the sequence.
- N: The k th term is any term in the sequence which provides us with a general solution or answer for all the terms in the sequence.
- T: k th / n th would be the answer of T_k ; helps to determine answer for any given position.

L's answer above was concise and clear and showed a definite understanding of the variables in the general term. However, K seemed to confuse k with T_k in the first part of her answer. Students were able to derive the formula for the general term of an arithmetic sequence and felt that it was more effective than their general term for a Fibonacci sequence as it enabled them to easily predict the 100th term of a sequence. At that stage they were not aware that an explicit formula could be derived for the Fibonacci series and proved inductively. However, they were not very sure about how many variables were involved in the formula $T_n = a + (n - 1)d$ (largely because they did not know the difference in meaning between the words formula and expression). Below follow their responses to question 8 of Questionnaire 2:

8.1 How many variables are there in the formula for the general term of an arithmetic sequence?

8.2 Why are the different variables necessary in the general term of an arithmetic sequence and what do they each represent?

- C: a - the first term.
 d - common difference.
 k - the position of the term.
- J: so that the general term can be used to work out several differing sequences by changing the variables a and d .
- K: a - Each one starts at a different number.
 d - the difference between each term differs.
 k - each time you are working out a different term.
- L: a - first term.
 d - common difference.
 k - no. of the term. They are necessary to represent the different aspects of the sequence.
- N: a - would represent the first term.
 d - would represent the common difference.
 k - would represent which term.

This suggests that most of the students regarded a , d and k but not T_k as being the significant attributes. They did not realise that a formula consists of all terms on the left and

right of the equal sign. Once again K seems to regard k as representing the actual term instead of T_k .

In Questionnaire 2 students were questioned regarding the usefulness of the general term of a geometric sequence.

9. Do you think it is useful to find the general term of an arithmetic sequence? Explain.

- C: Yes because it saves you time when having to work out a large value.
- J: Yes so that we can easily work out the value of for example the 100th term.
- K: Yes, helps you to find any other answer you need.
- L: Yes, then it is possible to predict any term in the sequence.
- N: Yes, it allows calculations to be done for any term.
- T: Yes, you can now determine the sequence by using a calculation instead of having to count.

The above responses were very positive, showing that they felt that the general term was important. This was reflected in their answers to the following question in Questionnaire 3, although C was not correct and T's response was badly expressed.

3.1 What is the use of a general term of a sequence?

- C: To work out the position of the term T_k .
- J: So that we can easily work out terms e.g. T_{103} .
- K: Helps you to find any term in the sequence which you want to find.
- L: In order to find any term in a sequence.
- N: General Terms helps to find any other term in the sequence.
- T: To work out any position in the sequence.

Finally, after the lessons had all been completed, students were asked the following question in Questionnaire 7.

1 Is it necessary to establish rules in mathematics? Explain.

- C: Yes, it makes it easier to work out problems
- J: Yes! It simplifies future working out.
- K: Yes, make it easier to work out answers.
- L: Yes, makes life easier.
- N: Yes, it makes life easier so that we won't have to waste time.
- T: Yes, we can deduce formulae and shortcuts to understanding maths.

The positive responses here all seemed to suggest that students had found the emphasis on generalisation had helped them to rise from level 0 to 1. The subsequent tables of ratings and graph suggest that generalisation had played a significant role in the rise from level 0 to 1.

Generalisation : the progress from level 0 to level 1						
Questionnaire	number	C	J	K	L	T
1	6	5	5	5	5	3
2	5.1 / 5.2	3	5	5	5	5
2	6	5	5	2	5	4
2	8.1 / 8.2	4	2	4	4	2
2	9	5	5	5	5	5
3	3.1	1	5	5	5	2
7	1	5	5	5	5	5
Ratings		1	2	3	4	5
number of responses		1	4	2	6	29

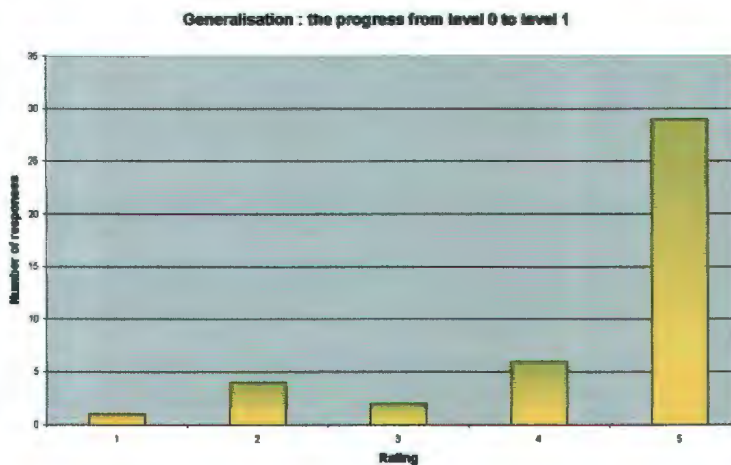


figure 42

5.4.2 The progress from level 1 to level 2

Further emphasis was placed on generalisation to facilitate the rise of students from level 1 to 2. Some of the activities associated with level 2 include stating definitions accurately and comparing different types of sequences. Thus students were asked to consider the following question in Questionnaire 2.

2. What different properties do all sequences have in common?

C: They are written, listed in a certain order and follow a certain pattern.

J: They are all ordered lists for which a formula can be devised to help predict the next term in the sequence.

K: They are all ordered lists of which the formulas alter - therefore resulting in different answers.

L: All sequences are ordered lists.

N: They follow a pattern.

T: Has order but also a listed pattern which can be predicted and calculated.

Thus students had noticed similarities possessed by sequences and could define an arithmetic sequence as illustrated by their responses to question 7.1 of Questionnaire 2.

7.1 What is an arithmetic sequence?

C: A sequence that increases or decreases by a constant base.

J: A sequence that increases or decreases by a fixed value

K: A sequence that increases or decreases by a fixed amount

L: It is a sequence that either increases or decreases by a constant amount

N: It is a sequence that inc / dec by a fixed value

T: A sequence that increases or decreases by a constant amount

To further test their understanding of different variables in a sequence formula and to compare different types of formulae, students were asked the following questions:

10.1 Explain the difference between $T_{20} = r$ and $T_r = 20$.

10.2 Find an example of a sequence for which

10.2.1 $T_{20} = r$

10.2.2 $T_r = 20$

C: 10.1: The 20th term = r in $T_{20} = r$
and in $T_r = 20$ the r th term = 20.

10.2.1 $T_{21} = 2r$ $T_{22} = 3r$

10.2.2 $T_1 = 20$ $T_2 = 20$ $T_3 = 20$

J: 10.1: In the first place, the 20th term = r
In the second place, the r th term = 20.

10.2.1 $r - 19$; $r - 18$; $r - 17$; $r - 16$;

10.2.2 20; 20; 20;

K: T_{20} tells us that the answer to the 20th term is r .

T_r tells us that the r th term is 20.

10.2.1 $r - 19$; $r - 18$; $r - 17$;

10.2.2 20; 20; 20;

L: $T_{20} = r$ means that the 20th term of the sequence is r ,
while $T_r = 20$ means that the r th term of the sequence is 20.

10.2.1 $T_1, T_2, \dots, T_{19}, r, T_{21}, \dots$

10.2.2 $T_1 = 20$ $T_2 = 20$ $T_3 = 20$

N: $T_{20} = r$ Term 20 has value r .

$T_r = 20$ Term r (variable) is always a fixed value 20.

10.2.1 $T_1 = r$ $T_2 = r$ $T_3 = r$

or $T_{19} = 0$; $T_{20} = r$; $T_{21} = 2r$; $T_{22} = 3r$;

10.2.2 $T_1 = 20$ $T_2 = 20$ $T_3 = 20$

The answers to 10.1 indicated that all students seemed to have arrived at a sound understanding of the meaning of the different variables in the formulae and their examples in 10.2 substantiated this viewpoint. Unfortunately, K's examples were the same as J's so in that case it is not certain whether or not they were original ones. In general, the answers to 10.1 and 10.2 show an understanding of the different variables in formulae as well as an ability to compare different types of formulae.

Students were then required to perform the second level activity of defining significant terms.

This was successfully done as indicated by the following responses to Questionnaire 3 nos

1.2, 1.3, 1.4 and 1.5.

1.2 What is a sequence?

All students replied that it was an ordered list.

1.3 What is an arithmetic sequence?

- C: It increases or decreases by constant value.*
- J: The terms differ by a fixed ratio*
- K: Increases or decreases by a constant amount per term.*
- L: One that changes by adding or subtracting a given value.*
- N: Ordered list include by fixed amount.*
- T: A list that goes up or down by a constant amount.*

1.4 What is a geometric sequence?

- C: Increases by a constant value (a multiple)*
- J: The terms differ are to a fixed ratio*
- K: Increases or decreases by a constant multiple.*
- L: One that changes by multiplying by a fixed number.*
- N: Ordered list include by multiplying / dividing.*
- T: Terms differ by a constant multiple.*

1.5 What is a Fibonacci sequence?

- C: The first 2 terms are given the next terms are obtained by adding the previous 2.*
- J: The two previous terms are added to produce the term in question.*
- K: The first two terms must be fixed and then you add the two previous to get the next term.*
- L: A sequence whose first 2 terms are fixed, and to get the next terms you add the previous 2 terms.*
- N: The first two terms are fixed for the third you add the first two.*
- T: When the two previous terms added give you the next term.*

Students were required to compare different types of sequences and statements written in reverse order from each other in the following questions 2.1 and 2.2 in Questionnaire 3.

2.1 Are all sequences arithmetic, geometric or Fibonacci? Explain.

- C:*
- J: Yes.*
- K: Yes*
- L: No, when you square root or square the numbers, then it is not changing by a fixed value.*
- N: No, squares and roots are made use of to create sequences.*
- T: Yes.*

2.2 Are all arithmetic, geometric and Fibonacci sequences examples of sequences?

Explain.

- C: Yes because each can be worked out and they have an order which follows each other.*
- J: Yes, due the fact that they are all called sequences quite aptly.*
- K: Yes, because for all of them you could work out what the next answer is.*
- L: Yes they are all ordered lists.*
- N: Yes, they follow an order/pattern.*
- T: Yes, you can use a formula to work out any given term.*

Only two students could think of perfect squares or square roots as other examples of sequences which had not as yet been considered under the titles of arithmetic, geometric or Fibonacci sequences.

The students were asked to write down and compare the general formulae for arithmetic and geometric sequences, noting any similarities or differences. All correctly wrote down the formulae and gave the following responses in questions 3.2 and 3.3 in Questionnaire 3..

3.2 Write down the general terms for arithmetic and geometric sequences.

- C: $T_k = a + (k-1)d$ $T_k = a(r)^{k-1}$
 J: $T_k = a + (k-1)d$ $T_k = a(r)^{k-1}$
 K: Arithmetic = $a + (k-1)d$ Geometric = $a(r)^{k-1}$
 L: $T_k = a + (k-1)d$ $T_k = a(r)^{k-1}$
 N: $T_k = a + (k-1)d$ $T_k = a(r)^{k-1}$
 T: Arithmetic $T_k = a + (k-1)d$ Geometric $T_k = a(r)^{k-1}$

3.3 Compare the general terms for arithmetic and geometric sequences, noting any similarities or differences.

- C: Geometric has $a(r)$ and the $k-1$ is at the top, the a and r are multiplied together. Arithmetic has a , d and the $k-1$ is at the bottom. The a and $(k-1)d$ are added.
 J: All involve $k-1$ and a being the first term in the sequence.
 K: Both have $k-1$, one is a power the other a multiple. The first term is added in the arithmetic whereas in geometric it is multiplied.
 L: Both have a and $k-1$.; Arith - (+) (d). Geom - (x) (r).
 N: Both have the first constant term. r = common ratio and d = common difference.
 T: Both have $(k-1)$, one is two variables multiplied, the other is added.

When the students wrote the general formula for both arithmetic and geometric sequences, writing a for the first term in each case and q for the constant differences for the arithmetic sequences but q for the common ratio for geometric sequences, J noted that for their formula $T_k = a + (q-1)k$ and $T_k = aq^{k-1}$: a and q are both bases; $k-1$ is a base in the arithmetic sequence and in the geometric sequence $k-1$ is an exponent. Below are recorded their responses to Questionnaire 3 nos 4.1, 4.2 and 4.3.

4.1 Write down the general term of an arithmetic sequence with the first term a and common difference q .

J, K, N and T wrote down $T_k = a + (k-1)q$

while C and L wrote $T_k = a(k-1)q$ (where the omission of "+" could have been careless, considering their previous responses).

4.2 Write down the general term of a geometric sequence with first term a and common ratio q .

All students correctly wrote $T_k = aq^{k-1}$, $T_k = a(q)^{k-1}$ or $T_k = aq^{(k-1)}$.

4.3 Compare your answers in 4.1 and 4.2 carefully, noting any similarities or difference.

- C: Same as 3.3 excepts use q instead of r and d .
- J: a and q are both bases and $k - 1$ is a base. In A sequence and in G sequence $k - 1$ is an exponent.
- K: Same as 3.2 and 3.3.
- L: Same as 3.3
- N: Addition
- T: In 4.1 q is multiplied by $(k - 1)$
In 4.2 $(k - 1)$ is to the power of q .

Students were required to compare sequences and series. They did have a good idea of the difference between the two but generally described a series as being a sum rather than the indicated sum of the terms of a sequence.

1. What is the difference between a sequence and a series?

- C: A sequence is an ordered list; A series is adding the terms of the series.
- J: A sequence is an ordered list where a series is the sum of an ordered list of terms.
- K: A sequence is a list of numbers; A series is the sum of the sequence.
- L: Sequence - list of terms; Series - sequence added up.
- N: A sequence is an ordered list while a series is the sum of all the terms in a sequence.
- T: Sequence is the list; Series is the sum of the numbers in the list.

Questions involving comparing general terms for geometric series followed and most of the students were successful here.

3. Are $T_k = 2 \cdot 3^{k-1}$, $T_k = 6^k$ and $T_k = 5 \cdot 7^{-k}$ all general terms for geometric sequences?

- C 1: Fits the formula. 2: Yes if it had been simplified. 3: I'm not sure.
- J: Yes, all can be simplified to the known general term of geometric sequences.
- K: Yes, all can be made into the original formula.
- L: Yes, they all increase by a power of r .
- N: Yes, there is always a coefficient a and r = base to the k .
- T: Yes they have powers.

When asked to define a limit, the students revealed that they had formed a reasonable concept of what it was in Questionnaire 5 no. 1.1

1.1 What is the limit of a sequence?

- C A fixed value.
- J: The final value of T_∞ which can be $+\infty$ or $-\infty$ or any rational number.
- K: A fixed value which it is heading to.

L: The highest / lowest no that any term in that sequence tends to.

N: ∞ ; $-\infty$

T: the number the sequence tends to.

Numerous questions have been included here to show the effect of stressing generalisation on rising from level 1 to 2. The large number of questions asked here and the high ratings achieved give an indication of the importance of generalisation in rising from the descriptive to theoretical informal level. The below graph suggests that generalisation plays a significant role in rising from level 1 to 2.

Generalisation : the progress from level 1 to level 2						
Questionnaire number	C	J	K	L	N	T
1 1.2	5	5	5	5	5	5
2 2	5	5	5	4	3	5
2 7.1	5	5	5	5	5	5
2 10.1	5	5	5	5	5	4
2 10.2.1	4	5	5	5	5	1
2 10.2.2	5	5	5	5	5	5
3 1.2	5	5	5	5	5	5
3 1.3	5	5	4	5	5	5
3 1.4	2	4	4	5	4	4
3 1.5	5	5	5	5	5	5
3 2.1	1	1	1	5	5	1
3 2.2	5	5	5	5	5	5
3 3.2 / 3.3	5	3	5	5	4	4
3 4.1/4.2/4.3	4	4	5	3	3	4
4 1	4	4	4	4	4	3
4 3	3	5	5	4	5	4
5 1.1	3	5	4	4	1	4
Ratings						
number of responses						
	6	1	7	23	65	

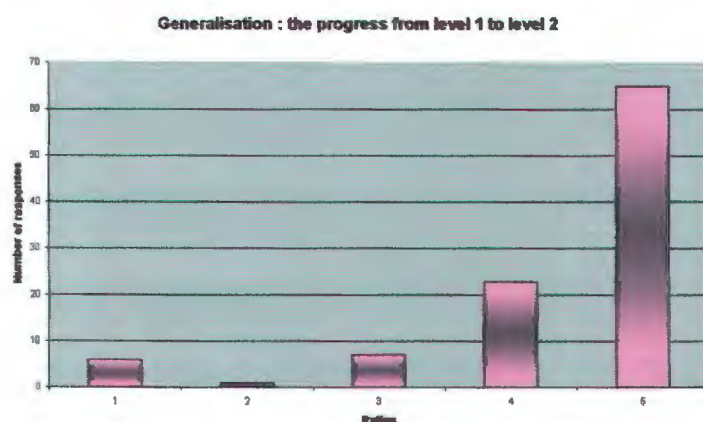


figure 43

5.4.3 The progress from level 2 to level 3

Once again generalisation was encouraged to promote the rise from level 2 to 3. One of activities associated with level 3 is using information about sequences and series to deduce more information. This needed to be done in the answering of questions 5.5 and 6 below. Students observed that there were infinitely many sequences which are both arithmetic and geometric such as 7, 7, 7, However, students lost ratings for stating that "any real

number" instead of stating that the general term is any real number and also for omitting to state that there would be infinitely many such sequences. They believed that $T_k = 0$ would lead to a sequence which would be arithmetic, geometric and Fibonacci but did not explain their answer nor questioned a constant ratio of $\frac{0}{0}$ being undefined.

5.5 How many sequences are there that are both arithmetic and geometric? Explain.

C wrote "An infinite amount, any real number." but all the other students stated it was "Any real number".

In questionnaire 4 students were asked questions regarding infinite sums before they had actually dealt with them together or thought about them before so they were unable to answer the questions very well as this was still something quite new to them at the time.

5.1 Do you think it is ever possible to evaluate $\sum_{k=1}^{\infty} [a + (k-1)d]$? Explain.

C Probably, I have not learnt it yet.

J: No, because the value is ∞ .

K: No, because it goes onto ∞ .

L: I don't know how.

N: No, the last number is un-identifiable.

T: No, there are too many variables.

5.2 Do you think it is ever possible to evaluate $\sum_{k=1}^{\infty} ar^{k-1}$? Explain.

C Probably, I have not learnt it yet..

J: No..

K: No because it goes on to infinity.

L: I don't know how.

N: No, you will never find the last term.

T: No, too many variables.

Students were then required to use their knowledge of sequences and limits to deduce answers. Their responses to questions 1.2 and 1.3 of Questionnaire 5 are recorded below:

1.2 Give an example of a sequence whose terms approach infinity.

C 3; 6; 9; 12;

J: $T_k = 3k$

K: 0, 0, 0, ...

L: 3 + 9 + 27 +

N: 2; 4; 6; 8;

T: 2, 4, 8, 16.

1.3 Give an example of a sequence whose terms approach a fixed numerical value.

C: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

J: $T_k = \frac{1}{k}$

K: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

L: $144 + 48 + 16$

N: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

T: $2; 2; 2; 2; \dots$

These previous answers all showed understanding and insight with the exception of K stating that 0, 0, 0 ... approached infinity instead of 0 and L's writing of answers as series rather than sequences.

Students were questioned about deducing more information regarding the infinite sums of arithmetic and geometric series. They realised that infinite sums of arithmetic series do not generally converge to a point value but did not mention the exception of $a = 0$ and $d = 0$. L and T did mention $d = 0$ but not $a = 0$ as well. In question 5, T gave an excellent answer but C did not know how to use absolute value properly and others gave incomplete answers when they said only between -1 and 1 without stating what only lay between -1 and 1. Nevertheless, they all seemed to realise that different cases needed to be considered for geometric sequences.

4. Does the sum of terms of an arithmetic series approach a limit? Explain.

C: No, it carries on into infinity. For it to be a limit it must be of fixed number..

J: No, as $\pm\infty$ is not a fixed value..

K: No, it always approach ∞ or $-\infty$

L: No, only the exception of 0 being the common difference.

N: No, ∞ , $-\infty$

T: No, only when $d = 0$.

5. Does the sum of terms of a geometric series approach a limit? Explain.

C: yes $-1 \leq r \leq 1$.

J: Some do and some don't. e.g. $T_k = \frac{1}{2^k} \therefore \text{limit} = 0$;

$T_k = 2^k \therefore \text{limit} = \infty$ which isn't a limit.

K: Yes, but only between 1 and -1.

L: Only if $|r| > 0$.

N: Yes, only between -1 and 1.

T: Yes.

Students were questioned regarding the need for proof and all were convinced of the need for proof. Some of the students did not supply a general reason for this, stating that k might not be an integer because they were thinking of particular examples they had done. Their

responses to question 3 of Questionnaire 6 follow below.

3. Do you think it is necessary to prove a statement regarding positive integers if you have shown it is true for about 10 values? Explain.

- C: Yes, because maybe the k th value isn't an integer.
- J: Yes, because maybe the k th value isn't an integer.
- K: Yes, because maybe the k th value isn't an integer.
- L: Yes, there might be a value for which the statement doesn't work i.e. a high value than 10.
- N: Yes, because it might not be true for one of the numbers in the line.
- T: Yes, it might not work for $11 \rightarrow$.

Questions regarding proof by mathematical induction followed at this stage. With the exception of at times writing assume $n = k$ instead of the statement being true for $n = k$ and N stating prove $k + 1$ instead of prove true for $n = k + 1$, this part was done well though they did not give a type of situation where it could be used.. They were very positive about the mode of proof by mathematical induction and found it to be both acceptable and logical. This may be seen in the following responses to questions 4.1 and 4.2 of Questionnaire 6.

4.1 How do you perform proofs by mathematical induction and when is it used?

- C: First prove that $n = 1$ is true then, Assume true for $n = k$ then prove true for $n = k \pm 1$.
- J: We prove that $n = 1$ is true, then assume that $n = k$ is true, then prove it is true for $n = k + 1$.
- K: First prove for $1 = n$ is true, then assume true for $n = k$, then prove true for $n = k + 1$.
- L: Prove $n = 1$, assume $k = \dots$, prove $k + 1 = \dots$. To prove a statement.
- N: First prove for $n = 1$, assume true for $n = k$, prove true for $n = k + 1$, \therefore True for all n .
- T: Prove $n = 1$, assume $n = k$, then prove true $n = k + 1$ \therefore formulae will work for any n .

4.2 Do you think mathematical induction is an acceptable method of proof? Explain.

- C: Yes because its logical.
- J: Yes, it is complicated and logical.
- K: Yes it is logical.
- L: Yes, it can be applied all the time.
- N: Yes, you prove easily, true for all n .
- T: Yes, it works to prove formulae.

At the conclusion of the lessons, students were questioned about the need to prove results in mathematics. The below responses to question 2 of Questionnaire 7 show that they were very positive regarding the need for this.

2 Is it necessary to prove results in mathematics?

- C: Yes, otherwise you would not know if the formulae worked.
- J: Yes, otherwise one can't be sure that results are true.
- K: Yes, it will show you understanding of the problem more clearly.
- L: Yes, otherwise it could just be lucky.
- N: Yes, you could have cheated and the person marking gives you 0.

T: Yes, so that people can know that we know what we're doing.

The responses to the above questions show students had definitely become aware of the need to generalise and prove results. After having only a few lessons, they had really come to appreciate the idea of finding rules to speed up their work and were in a great hurry to find them. They readily accepted proof by mathematical induction (which is not normally taught to matriculation students). The numerous number patterns studied, the provision of examples to show that many statements are true but may not always be true as well as the domino illustration all seemed to have made contributions in these areas.

Below is the graph regarding advancement in thought levels from level 2 to level 3. The majority of ratings being 3, 4 or 5 indicate that generalisation had made a strong impact in this regard. The fact that there were more threes than usual here stems from the fact that several questions did not have yes or no answers with a reason but rather types of questions where 3 could be given because of some omission in the answer.

Generalisation : the progress from level 2 to level 3						
Questionnaire number	C	J	K	L	N	T
3 5.5	4	3	3	5	3	3
3 6	1	4	4	4	4	4
4 5.1	1	4	4	1	4	1
4 5.2	1	1	2	1	2	1
5 1.2/1.3	5	5	3	4	5	5
5 4	3	4	4	4	4	4
5 5	3	5	3	2	3	1
6 3	3	3	3	5	5	5
6 4.1	4	3	4	3	4	3
6 4.2	5	5	5	4	5	5
7 2	5	5	4	5	3	4
Ratings	1	2	3	4	5	
number of responses	9	3	16	21	17	

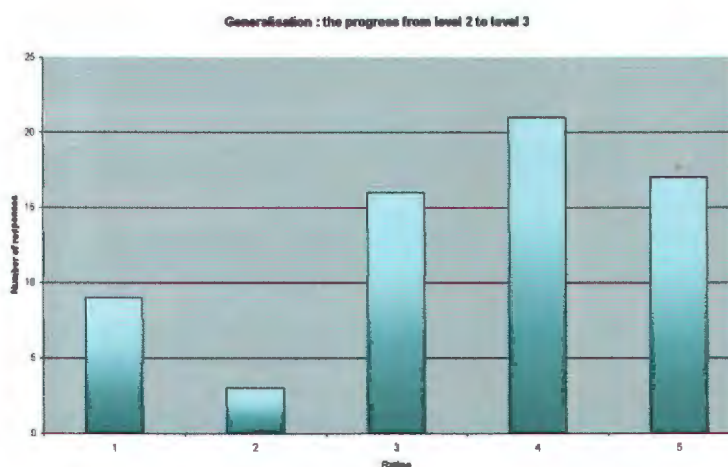


figure 44

5.4.4 General

Another general type of question asked in the questionnaire was to make summaries of work done. This is something advocated by van Hiele. The example below indicates reasonable but rather brief attempts because students seemed to be in too much of a hurry because of time limitations.

10. Give us a summary of everything we have done so far.

C: Fibonacci Sequence, Geometric Sequence, Arithmetic Sequence, Patterns.

J: Expand Sequence and worked them out.

K: Pattern, Sequences, Arithmetic, Geometric.

L: We have looked at different types of pattern and then sequences. We looked at Fibonacci, Arithmetic and Geometric Sequences.

N: Patterns, Sequences, Arithmetic, geometric and Fibonacci.

T: Learnt various sequences, learnt how to identify patterns.

The overall results of all the questions asked regarding the impact of generalisation on progress through thought levels are summarised in a table and the graph is indicated thereafter. The graph below covers all questions from levels 0 to 1, 1 to 2 and 2 to 3 as well as the general question 10 here. It seems to indicate that generalisation was relevant in this regard.

Generalisation : General							
Questionnaire number		C	J	K	L	N	T
3	10	4	2	3	4	3	3
Overall Ratings		1	2	3	4	5	
number of responses		16	9	28	52	111	

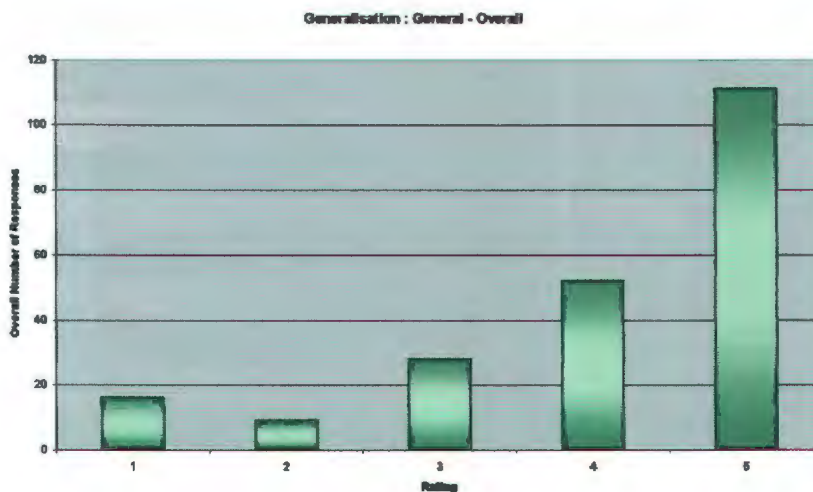


figure 45

5.5 Conclusion

Finally the students indicated that they enjoyed working together, discussing what they were doing, making generalisations and proving things for themselves. Their responses to questions 6.1 and 6.2 of Questionnaire 7 below give an indication of this:

6.1 Do you think discussing problems with fellow students helps you to understand concepts? Explain.

- C: Yes, because they might understand more than you.*
- J: Yes! by seeing other's point of view we can get a broader view on the subject and approach it from different angles.*
- K: Yes, you can put all your ideas together to help you reason an answer.*
- L: Yes, they experience the same problems, and when they figure them out, they can explain it as well.*
- N: Yes, you can work as a whole group and a lot better if you know what's going on.*
- T: Definitely, often they can help you understand something even your teacher couldn't.*

6.2 Do you think establishing rules for yourselves gives you a better understanding of a topic in mathematics? Explain.

- C: Yes, because if you understand how the rule came about you can understand how to solve the problem.*
- J: Definitely. One fully understands and never forgets rules established by oneself.*
- K: Yes, by figuring it out yourself you can see clearly how the rule came about. You are not just presented with the rule and have to accept it.*
- L: Yes, it shortens the route to an answer, and makes understanding of an answer a lot simpler.*
- N: Yes, rules are there to make things easier and hence give you a better understanding.*
- T: Yes without them working out would be random and our understanding would be confused.*

The above responses indicate how students did appreciate working on their own to generalise and ultimately prove some results. They had advanced from level 0 right through to level 3 thinking in the topic of sequences, series and mathematical induction. Establishing rules for themselves had made them very conscious of generalising and they were always trying to find quick routes and discover rules as soon as possible. The general graphs of the impact of visualisation, exploring patterns and generalisation are indicated together hereafter. They all seem to reveal a definite positive effect on progress through the levels of thinking.

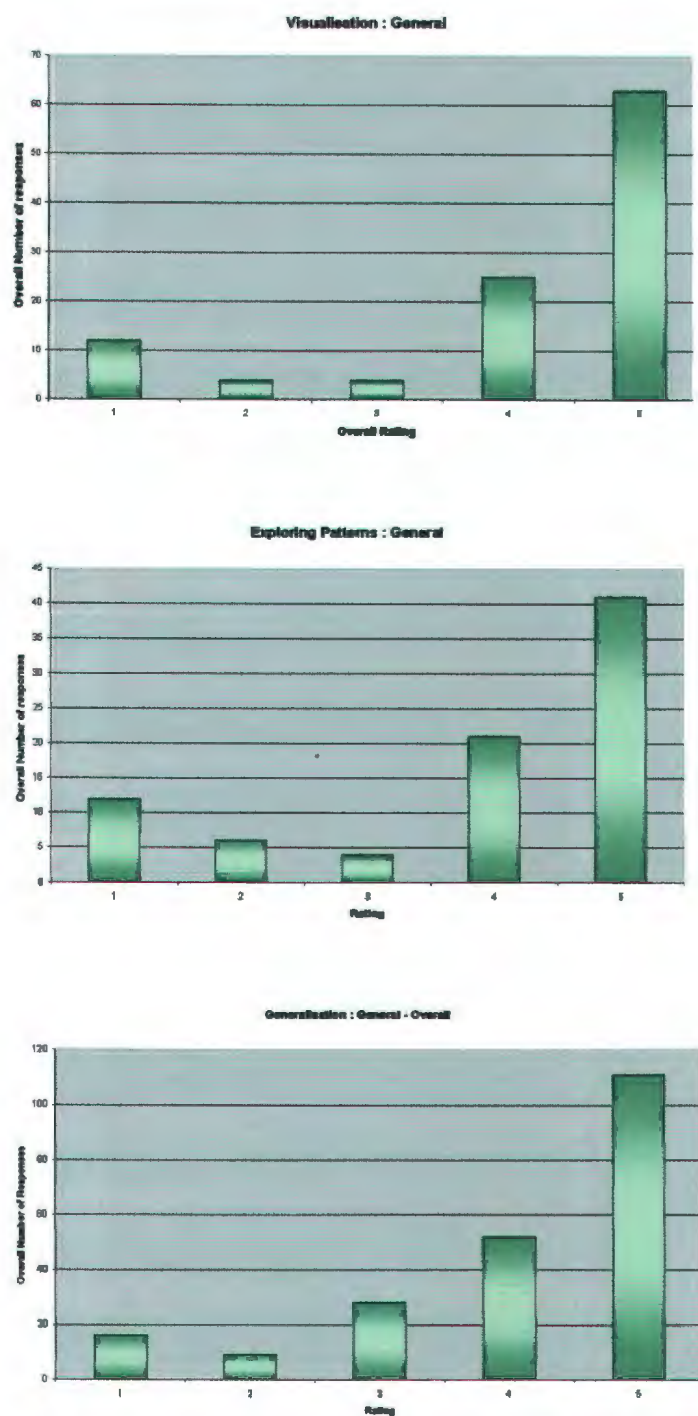


figure 46

The student responses to the questionnaires seem to give some indication of the beneficiary effect that the emphasis of visualisation, patterning and generalisation might have on the development of the thought levels of students. Implications from these responses as well as recommendations resulting from them will form the theme of the final chapter.

CHAPTER 6

Conclusion and recommendations

6.1 Introduction

The students' responses to the questions regarding sequences, series and mathematical induction in the questionnaires seemed to reveal some significant factors regarding the learning of these topics. These results were recorded and graphed in Chapter 5 under the headings of visualisation, exploring patterns and generalisation and will be analysed under the same headings in this chapter too. Although the group involved in the research here was small, nevertheless some valuable implications do seem to have emerged out of the study. As a result, conclusions will be drawn and recommendations made.

6.2 Visualisation

6.2.1 Introduction

Various visual illustrations were provided to assist pupils rise from levels 0 to 1, 1 to 2, 2 to 3. The effectiveness of these examples will be considered at each of the relevant stages.

6.2.2 The progress from level 0 to level 1

Visual examples were chosen in such a way that their visual appearance would reflect their properties with a view to later on, as advocated by van Hiele, letting the appropriate name itself reflect the relevant properties. According to the lessons taught and the comments of students involved, visually recognisable patterns were relevant in the establishment of concepts regarding sequences and series. They could appreciate the growing size of objects and were able to visualise further terms of the sequence in their minds. In some cases counting and calculation led to a realisation of the need for the n th term of a sequence. Others, however, could not see any connection between visual illustrations and the establishment of rules. Nevertheless, the overall response of students and the graph indicates that visualisation had helped to promote the rise from level 0 to level 1.

6.2.3 The progress from level 1 to level 2

Once rules had been established and students were operating on level 1 in a topic, they could not all appreciate how this would help them do algebraic examples in which the rules were utilised. This led to a discussion on what visualisation actually is and it was concluded that it means seeing with the mind's eye. Some students claimed that visualisation could help them see how the sequence would develop or picture what was happening in their own minds. Others, however, claimed that they could not see how visualisation could help them do certain examples. They were of the opinion that visualisation did not serve a purpose when working on a more abstract level with symbols.

Although students were of divided opinions regarding the relevance of visualisation in the performance of all level 1 activities, they were extremely enthusiastic regarding the use of visual examples used to help them rise from level 1 to level 2 to appreciate the summation formulae for both arithmetic and geometric series. They felt that it had provided them with a good base to work from and helped give meaning to the formulae derived on the second level. The block diagram for geometric series was successful when dealing with only the particular cases n equals 1, 2 and 3 but the formula for the n th case was derived from the pattern of the numerical results they had found rather than trying to interpret the diagram representing the n th case. This would definitely be a third level activity which seems to involve a type of interpretation tending to possibly be unfamiliar and not well appreciated by students.

The graph representing the impact of visualisation on progress from level 1 to level 2 seems to give a clear indication of the effectiveness of visualisation in the rise from level 1 to 2. Students generally seemed to feel that although they could not always directly link it to more abstract derivations and applications, it certainly gave them a sound basis from which to proceed.

6.2.4 The progress from level 2 to level 3

Various activities encouraging the emphasis of visualisation to facilitate the rise from level 2 to level 3 were organised. These were used to find sums of infinite series, limits of sequences and series and proof by mathematical induction.

Several types of visual representations were used to introduce the concept of a limit. The three types of visual illustrations used included many numerical values sometimes using calculators and primarily using computers, block illustrations and graphs. There was general consensus that computer values helped them to appreciate how terms or the sum of terms of a sequence can get closer and closer to a number.

Students were given various block illustrations to find the sums of infinite series such as:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots; \quad \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots; \quad \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots$$

They began to appreciate this type of visual illustration and were able to derive a formula for S_{∞} . However, these were the only series illustrated by block diagrams used as the case

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \dots$$

was omitted because of the difficulties certain students had been experiencing with block diagrams particularly representing a variable number n of divisions.

There were mixed feelings regarding the use of block diagrams to illustrate limits. Half of the students claimed that as the blocks were being shaded, they could see that the limit was one. However, others felt that they did not understand. Perhaps this was because they had not been provided with many opportunities of visualisation in the past or it could be that their minds did not operate or were not trained to work in this particular way. However, the graphical method was successful and positively accepted by the students. They felt that the computer method was the best and most acceptable one followed by the graphical method. The block illustration was not as popular but all pupils appreciated the provision of three different types of visual illustrations for the same concept. They felt that it deepened their understanding and helped to cater for their different types of minds.

Visualisation was also employed to help introduce the concept of mathematical induction. The domino illustration received very positive acceptance from the students. They all felt that it helped them to visualise what was happening and why the various steps in a proof by mathematical induction were necessary. They believed it helped them picture how there could be a breakdown in the chain and valued the provision of a practical illustration. The ladder analogy was also appreciated in this regard. The block illustration was done last because by this stage experience had shown that it was not very well received by students. Only half the students found that it helped them to appreciate the concept of proof by mathematical induction even though only a simple two dimensional example was chosen. Students found it very difficult to visualise the n th term of a series in a block diagram. Even though interpretation of visual illustrations is a fourth level activity, it seemed to introduce another complex aspect to the problem and often hindered rather than promoted their understanding. Fortunately, they clung to the domino and step ladder analogies which really helped them to appreciate the concept. Once again the graph depicting the impact of visualisation on progress from the theoretical informal to the theoretical formal stages was positive.

6.2.5 Conclusion

The provision of visual examples certainly did help establish a solid base for students in their formation of concepts. In most cases it seemed to be helpful in promoting elevation from each level to the next. Only some pupils recognised visualisation as forming part of their working with more abstract examples although they could see it forming part of their practical applications. Examples utilising computers, graphs and dominoes were very successful. Although block diagrams contrived to illustrate concepts were effective in establishing formulae for the sum of terms in an arithmetic series, they were not as effective in the cases of geometric series and mathematical induction. Some students found it difficult to visualise an n th term in a geometric diagram and at times it seemed to introduce an additional problem which clouded the issue rather than promoting their understanding. Consequently, as lessons progressed, less emphasis was placed on these sort of illustrations and more

emphasis was placed on practical, computer or graphical illustrations which were universally well received.

The graph representing the general effect of visualisation on progress from one level to the next suggests that visualisation plays a significant role in the development of mathematical thinking.

6.3 Exploring Patterns

6.3.1 Introduction

Students were encouraged to explore patterns to facilitate their rise from levels 0 to 1, 1 to 2 and 2 to 3. The effectiveness of this approach in the rise from one level to the next in each case will now be considered.

6.3.2 The progress from level 0 to level 1

Students first studied patterns before being introduced to sequences. They felt that patterns involved repetition, were necessary to produce a sequence and formed the building blocks for a sequence, which they recognised as being an ordered list of terms.

It was felt that patterns have a certain order so that the rest of the pattern which is not given can be worked out or calculated. The fact that some aspect of the pattern make it visible and showed some kind of repetition enabled subsequent terms to be predicted. However, some patterns were not as obvious as others and had to be studied more closely, in some cases requiring calculations, in order to establish the pattern.

They felt that patterns had helped them to understand the concept of a sequence. It helped them to recognise a pattern more readily, discover the order connected with the sequence and establish the last few terms of it. Furthermore, it often gave them the insight to create a formula for the n th term of a sequence. There was general consensus that the study of patterns had helped students understand the basics. It taught them to recognise patterns in

various situations and appreciate the relevance of a formula.

The graph in chapter 5 gave an indication that although very tricky patterns may not always lead to easy formulation of rules, on the whole exploring patterns had made a positive contribution to the rise from the pre-descriptive to the descriptive level.

6.3.3 The progress from level 1 to level 2

There were various indications that exploring patterns had helped to promote the rise from the descriptive to the theoretical informal level. It appeared that students had begun to realise how to extract the essence of patterns in order to formulate definitions and compare different patterns. They felt that they had become used to looking for patterns and detecting types of sequences by exploring patterns. Formulae too could be more easily derived by considering patterns leading to their establishment.

Initially students felt that exploring patterns helped them with worksheets based on the formulae they had established but when they arrived at the topic of geometric series, they did not all believe that patterns played a very big part in examples of a more abstract nature. The pattern in Pascal's triangle and its application for finding coefficients of terms in expanding binomial powers was very well appreciated. The students were most aware that the arrangement of numbers in a triangular form made it easier to detect and continue the pattern. This as well as other examples regarding number patterns to encourage them to search for number patterns and detect apparent results were given to lay a firm foundation for proof by mathematical induction at a later stage.

The graph regarding the importance of the role played by patterns in promoting the rise from level 1 to level 2 as perceived by the students is positive. It suggests that exploring patterns could be relevant in this regard.

6.3.4 The progress from level 2 to level 3

Various activities were organised to encourage the rise from level 2 to level 3. For example, students were encouraged to describe how Pascal's triangle was formed. This they were able to do but were not all able to give a complete description, sometimes forgetting to mention some aspect of the pattern. An argument based on a visual representation is a third level activity. The students were nevertheless very appreciative of the importance of the layout and application of the pattern in Pascal's triangle.

When students were asked whether patterns helped them to see the need to prove results in mathematics, some were positive but others felt that by studying a pattern they were not going to prove a result. Number patterns and apparent formulae for summations such as the sum of all odd and even numbers had been studied to make students conscious of general results arising from number patterns. An example true for the first forty numbers but not for the forty-first term make them very aware of the need for general proofs of apparent number patterns. Every student found proof by mathematical induction logical and their awareness of apparent number patterns and the need to prove results in general did seem to make some contribution to their rise to level 3 in this regard.

The graph depicting the rise from level 2 to level three does not seem to be as effective as the two previous pattern graphs. The lower ratings resulted from students not always perceiving a direct link between patterning and proof, even though it may have led them to the point where proof is seen to be necessary or relevant. The fact that there were more fours and fewer fives this time seemed to result from the students' incomplete descriptions of visual representations. Nevertheless, the fact that more than 70% of the ratings were 3 or more seems to indicate that patterns in fact could play a significant role in the rise from the theoretical informal to the theoretical formal level.

6.3.5 Conclusion

There seemed to be a general consensus amongst students that the study of patterns had helped to establish a firm foundation for the topic of sequences and series. As time progressed, they became very used to looking for patterns in order to establish rules as soon as possible. Although they did not all seem to perceive a direct link between patterns and proof, after studying numerous number patterns and after becoming aware that not all apparent patterns are generally true, they were certainly ready to accept proof by mathematical induction. The general graph depicting exploring patterns seems to suggest that exploring patterns is effective in the rise from each level to the next.

6.4 Generalisation

6.4.1 Introduction

As students were studying the topic of sequences and series, they were encouraged to make generalisations and the effectiveness of this approach at various levels was taken into consideration in the previous chapter.

6.4.2 The progress from level 0 to level 1

At level 1 students should be able to discover formulae involving symbols by experimentation as well as recognise and state algebraic properties. They should also be able to perform elementary manipulation of symbols.

Students were positive about the use of establishing the n th term of a sequence as it enabled them to work out any term without working out all of the terms of a sequence. They regarded it as being an easy way of processing numbers without having to go through all terms of the sequence or series.

They were enthusiastic about forming general rules. They noted that the general term of a Fibonacci sequence gave the rule but was not very satisfactory because it did not enable terms to be calculated easily as each term from term 3 onwards would depend on the

previous two terms. Initially all but one of the students recognised the difference between T_k and k in the general term of a sequence but in general they regarded the formula $T_k = a + (k - 1)d$ as having three variables because they did not recognise T_k as being a variable in the formula. When they came to differentiating between $T_{20} = r$ and $T_r = 20$, they all clearly showed that they understood the difference between the two statements.

Students revealed definite signs that they had risen from level 0 to level 1. They saw the need to establish rules to simplify their working out in the future. They felt that rules helped to make life easier because they eliminated the need to work out results term by term. The graph depicting the students' apparent rise from the pre-descriptive to the descriptive level in Chapter 5 gave an indication that emphasis on generalisation had played a significant role in their progress to level 1.

6.4.3 The progress from level 1 to level 2

Students were able to operate efficiently with level 1 type examples in which the formulae they had established were utilised. They regarded formulae as being important because they were deduced to create shortcuts, prevent wastage of time and promote their understanding of mathematics. They recognised the fact that the formulae they had established made it easier to work out problems on level 1.

After working with and discussing arithmetic and geometric sequences and series, students showed that they had risen to level 2 in their ability to define arithmetic and geometric sequences and series. However, occasionally they did lose some points in their ratings by forgetting to mention some aspect involved in a definition or procedure. Students showed the ability to compare different properties possessed by various types of sequences. They were able to recognise differences between them as well as formulate statements showing the interrelationship between symbols. They appreciated the level 2 activity involving the derivation of the summation formulae for arithmetic and geometric series. The graph relating to growth from level 1 to 2 as a result of emphasising generalisation seems to indicate that

generalisation does play an important role in the rise from the descriptive to the theoretical informal level.

6.4.4 The progress from level 2 to level 3

One of the level 3 activities attained by the students was to use information about sequences and series to deduce more consequences. They were able to formulate arguments based on diagrams or visual representations. After studying various cases of limits, they were able to grasp the concept of what it means to converge to a limit. Students studied various cases of convergence and divergence of series. This led successfully to the derivation of the formula $S_{\infty} = \frac{a}{1-r}$ for convergent geometric series. This was a third level activity as it involved deducing consequences from other information.

After studying numerous examples regarding number patterns, students concluded that certain results regarding sequences and series appeared to always be true whereas others were not necessarily so. They observed that even if a result appeared to be true for the first ten natural numbers, it need not always be the case. This made them aware of the need for the form of proof known as proof by mathematical induction.

The students provided several reasons for the need for proof by mathematical induction. These included knowing for sure whether or not a formula worked so that one could be certain whether or not general results would be true as well as using a mode of proof which would be universally acceptable. They were happy to accept proof by mathematical induction as a logical form of proof which had relevant applications in mathematics.

The graph relating to the impact of generalisation on the rise from level 2 to level 3 shows many ratings of 3,4 and 5. The fact that there are more 3 and 4 ratings than usual could be attributed to the fact that several questions did not require a response of yes or no with an explanation but rather some type of answer in which one or two relevant aspects being omitted or incorrect could result in some loss of ratings. The overall impression of this graph

does, however, suggest that emphasising generalisation is important in the rise from the theoretical informal to the theoretical formal level.

6.4.5 Conclusion

Students became enthusiastic about generalising results and, as the lessons progressed, they grew into the habit of seeking general results as quickly as possible. It became almost like a competitive game for them. They were positive about discussing problems with their fellow students as it promoted their understanding and gave them a broader picture of the topic at hand. They really appreciated the opportunity of generalising themselves instead of being given the end results as they felt that establishing results helped them to always remember them, clearly see how they came about and know how to apply them. They felt they had benefited from forming an active part in the learning process and being able to reinvent results for themselves.

They were encouraged to consolidate what they had learnt in a section by forming summaries as advocated by van Hiele. The final activity they performed at the conclusion of the lessons when they were asked to derive a formula from a pattern and then prove it for themselves by mathematical induction was very successful. It gave an indication of how the students had risen up through the various levels to reach the theoretical formal level. The graph relating to the impact of emphasising generalisation at all levels seems to indicate how effective it is in the learning process.

6.5 Recommendations

Even though only a small group of students were used in this study, they did represent a reasonable range of abilities and modes of thought of both male and female higher grade matriculation mathematics students. Some of the results which emerged from the study do seem to be relevant and have some significance in the teaching of sequences and series or other algebraic topics covered by matriculation students.

The following recommendations could be made in this context:

- The levels of thought of learners should be taken into account in the teaching of sequences and series to matriculation students.
- Visualisation should be encouraged in the teaching of sequences, series and mathematical induction. Visual illustrations used should be appropriate for the topic being taught and should not be so complex that they hinder rather than promote the understanding of the pupils.
- Patterning should be studied and emphasised in the teaching of sequences and series to enhance understanding and insight.
- Students should be encouraged to generalise results for themselves in sequences and series instead of always being provided with formulae at the beginning of a topic.
- By taking the various learning levels into account and providing opportunities of visualisation, patterning and generalisation to facilitate progress through these levels, proof by Mathematical Induction could be taught to higher grade mathematics matriculation students..
- Mathematics should be taught in such a way that students become actively involved in the learning process as advocated by Freudenthal.

Further research which could be relevant in this area could be:

- The influence of the van Hiele levels on other algebraic topics such as group theory and on other branches of mathematics such as trigonometry and analytical geometry.
- Teaching mathematics within a more intense socio-constructivist methodological framework.
- The influence of gender on the approach to teaching mathematical concepts.
- Larger emphasis on the development of understanding of proof.
- Higher technical emphasis on visualisation / modelling.

The influence of promoting visualisation, exploring patterns and generalisation in these areas would appear to be relevant. Furthermore, the effects of teaching more secondary school mathematics topics as what Freudenthal describes as a reinvention would be an interesting and relevant field of study.

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APPENDIX A

Worksheets used in empirical component



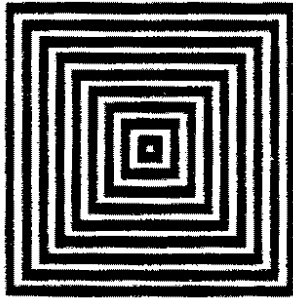
M.C. Escher in (Jacobs 1970: 19)

1. What is a pattern?
2. Give 2 examples of patterns.

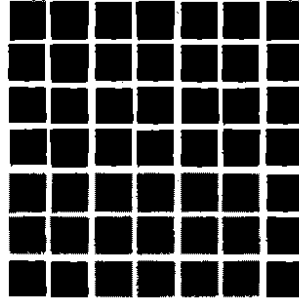
Worksheet 1.2 Patterns

1. Look at the following Patterns:

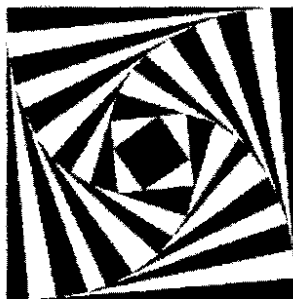
Here are some other examples of optical art.



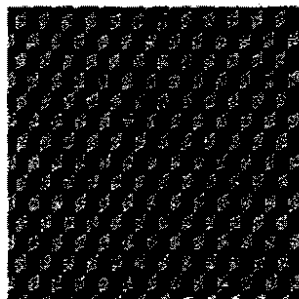
Example A



Example B



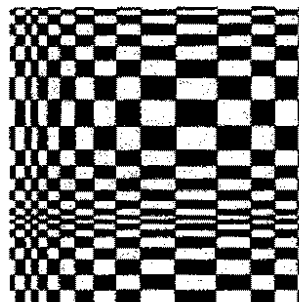
Example C



Arish quilt, repeating block design
Example D

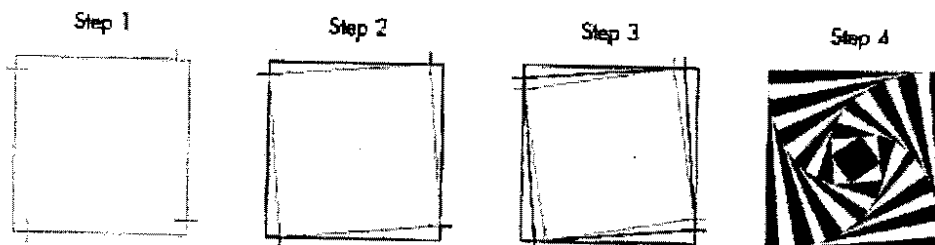


Japanese Op Art: Hayase Juchi, Dover Publications
Example E



Common Apposition, geometry student
Example F

2. Note the steps taken to arrive at example C.

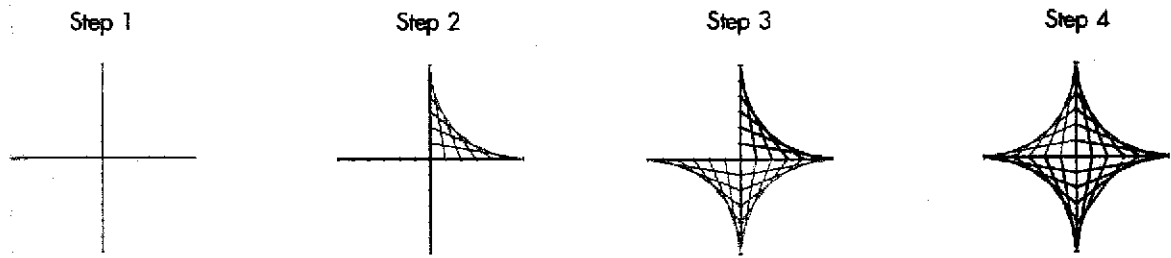


(Serra 1997: 14-15)

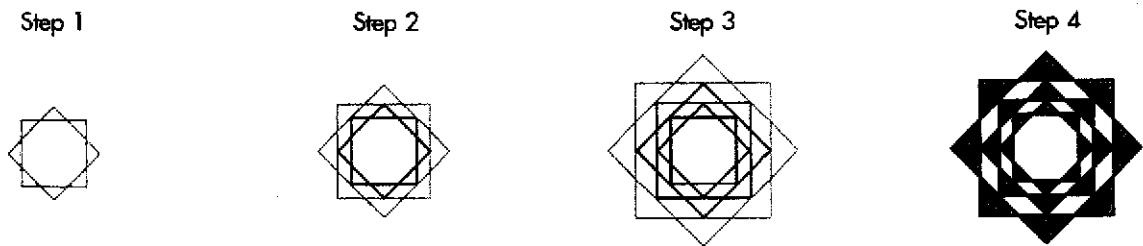
IV

3. Here are some more examples of steps followed to establish patterns.
Study them carefully and discuss them with each other:

3.1



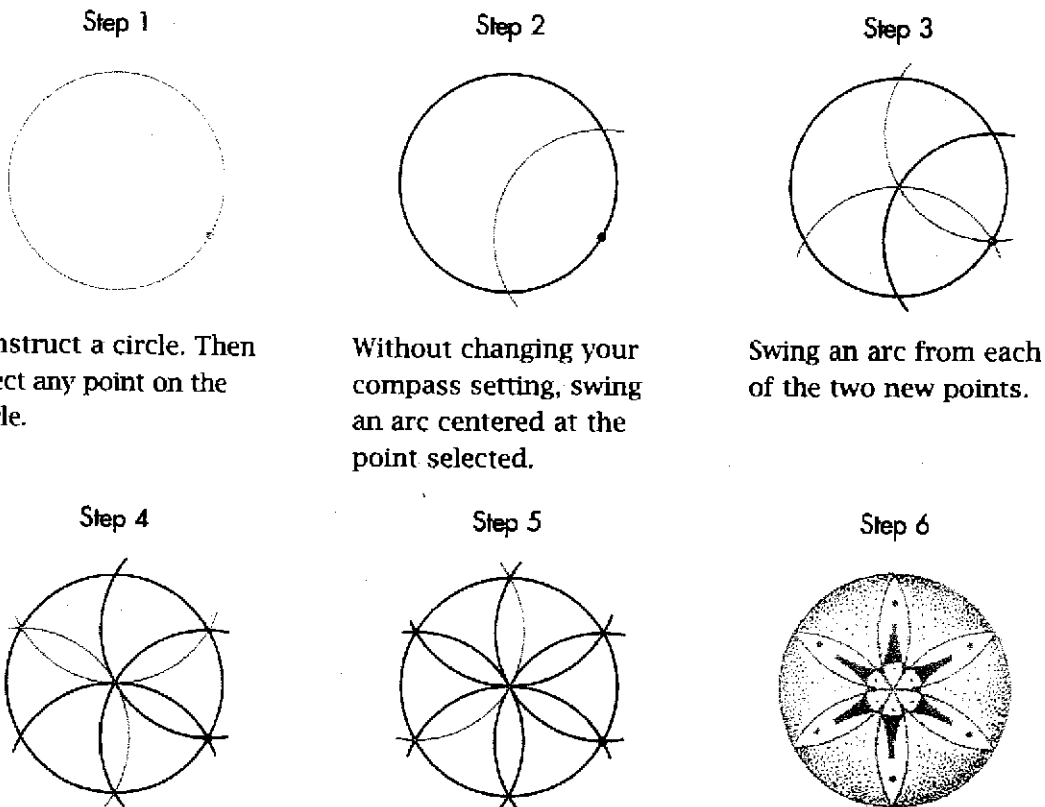
3.2



(Serra 1997: 7)

3.3

The compass is a geometric tool used to construct circles. You can make some very nice designs with only a compass. A daisy is one such simple design. The figures below give you the necessary instructions to make a daisy. Read through the steps for making the design before you begin the exercise set at the end of the lesson.



Construct a circle. Then select any point on the circle.

Without changing your compass setting, swing an arc centered at the point selected.

Swing an arc from each of the two new points.

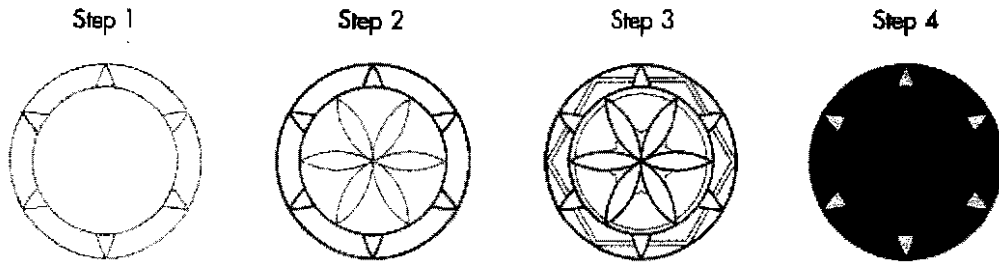
Again, swing an arc from each of the two new points of intersection.

Swing an arc connecting the last two points of intersection.

Decorate your daisy design.

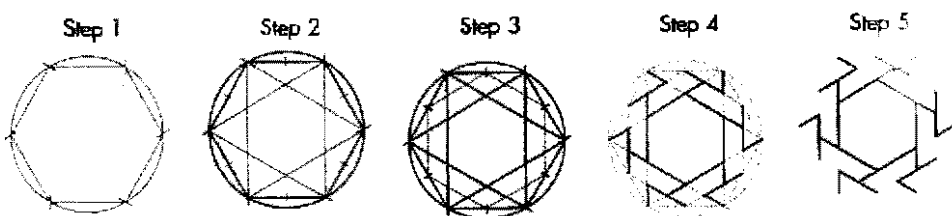
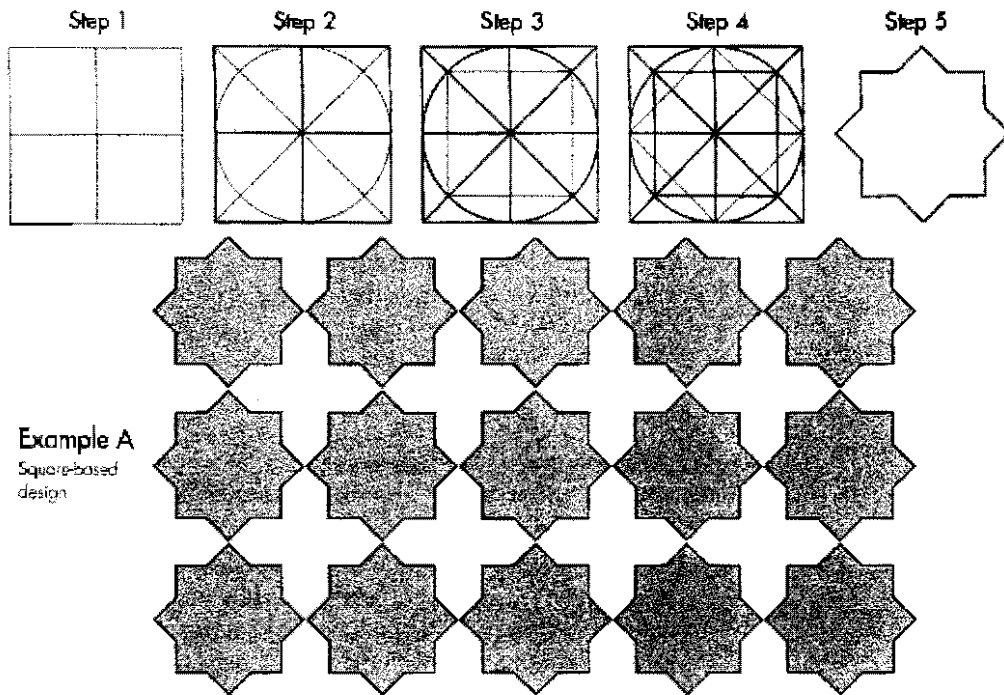
(Serra 1997: 11)

3.4

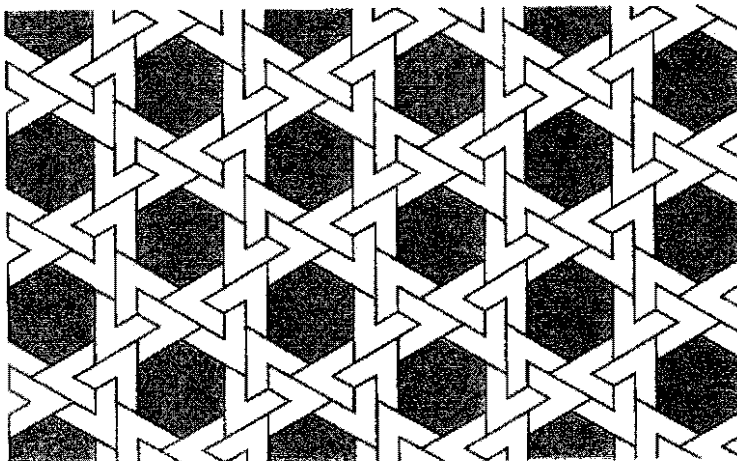


(Serra 1997: 18)

3.5

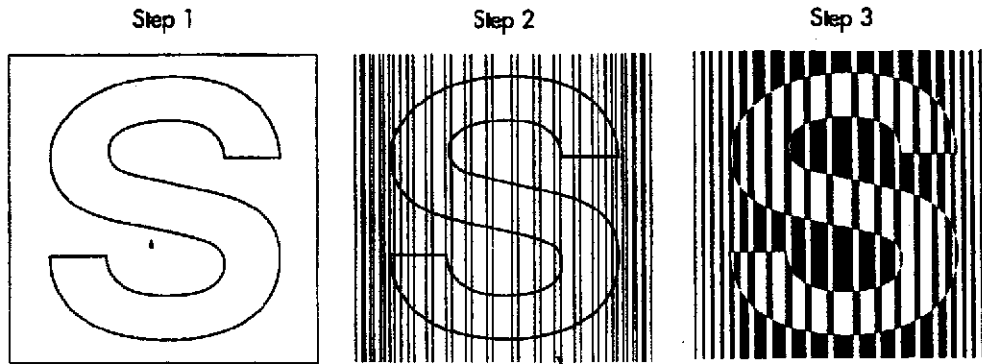


Example B
Hexagon-based design



(Serra 1997: 24)

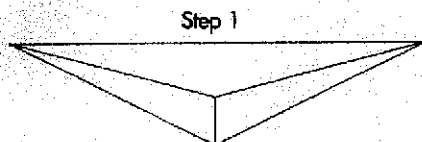
3.6



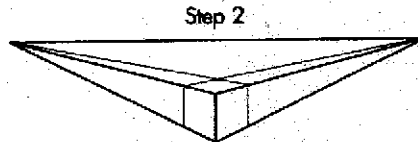
(Serra 1997: 13)

3.7 Drawing a High-Rise Complex

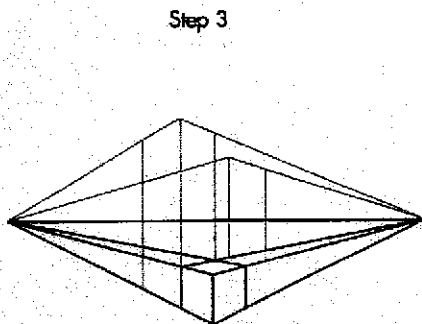
Skyscrapers are challenging to draw in perspective because they can be made of many different rectangular solids and thus have many different vanishing lines. Drawing a block of skyscrapers in two-point perspective is illustrated for you below.



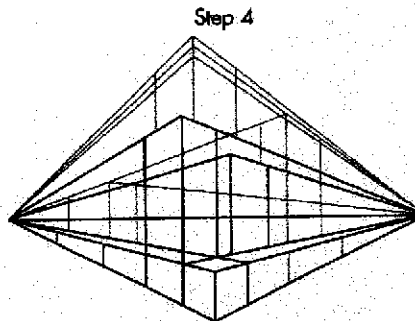
Begin with a horizon line and two vanishing points. Draw the front vertical edge of your first building with all the vanishing lines.



Complete the two-point perspective view of the first building.

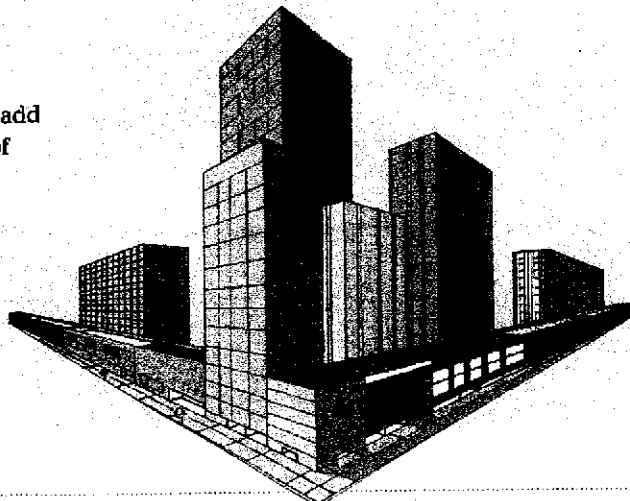


Next, draw in a couple of the taller buildings. Start with the front vertical edge of each building and draw the vanishing lines. Complete the perspective view.



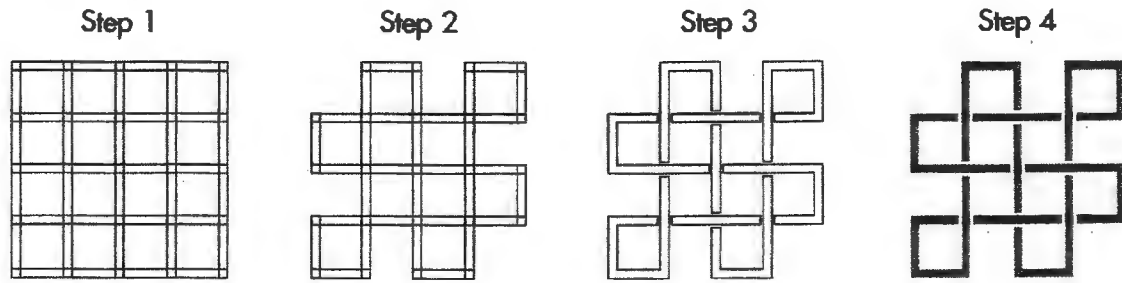
Create additional buildings and use vanishing lines to add architectural details.

Step 5
Erase all unnecessary lines and add other details. See the example of student art on the next page.

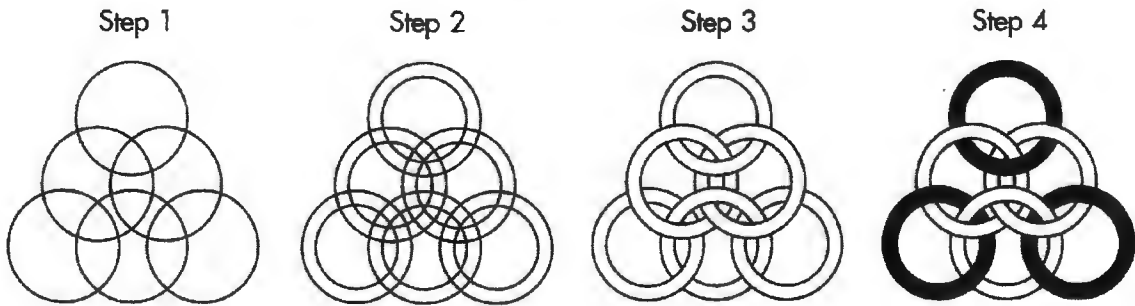


(Serra 1997: 32)

3.8



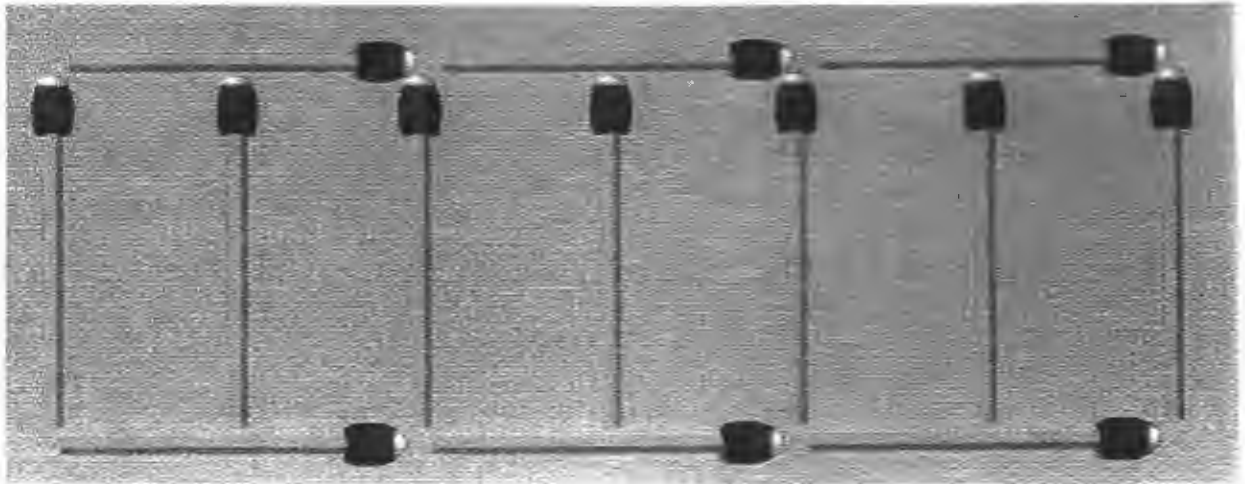
3.9 A similar approach can be used to create knot designs with rings.



(Serra 1997: 21)

4.

Below is a pattern made out of matchsticks. Use the matches with which you have been provided to create patterns.

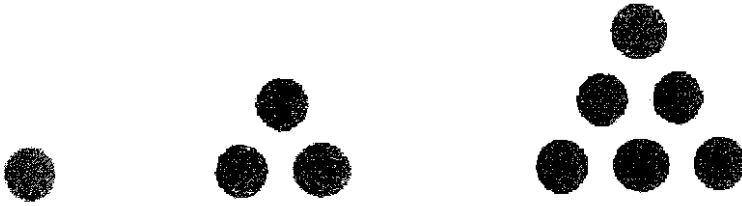


(Serra 1997: 33)

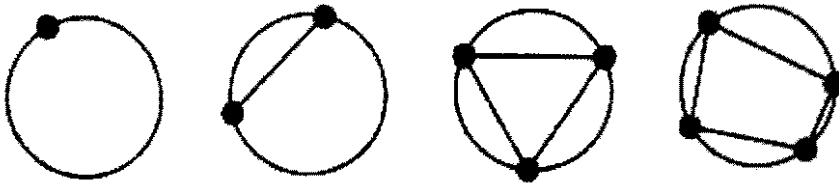
Worksheet 1.3 Patterns

1. Draw the next shape in each of the following picture patterns:

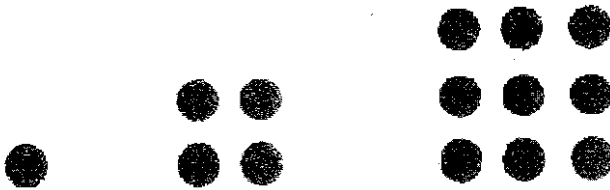
1.1



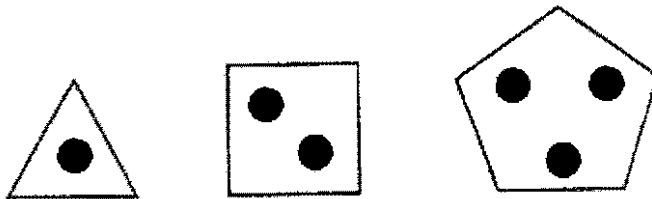
1.2



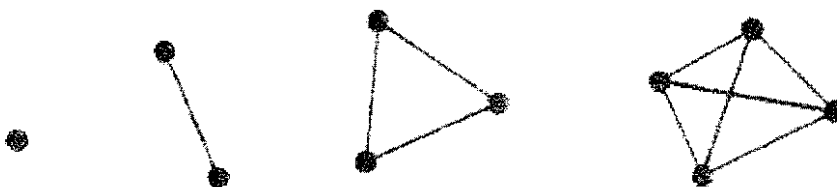
1.3



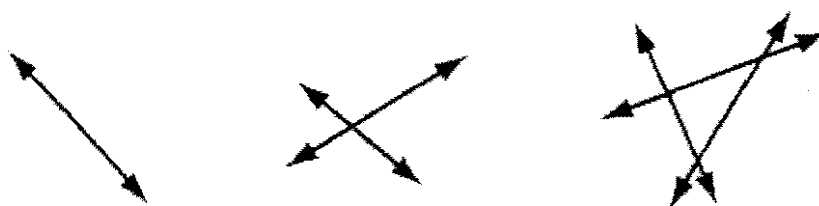
1.4



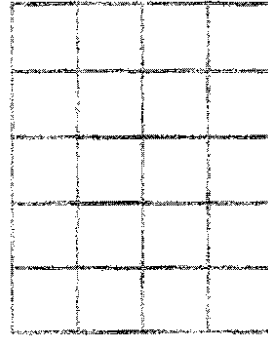
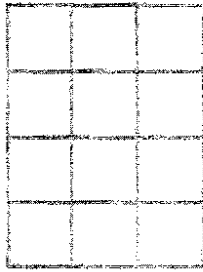
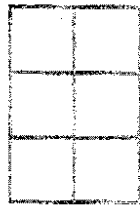
1.5



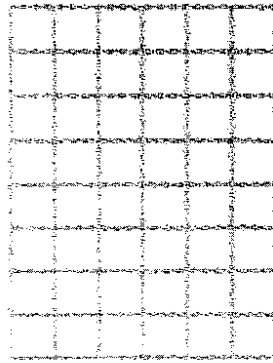
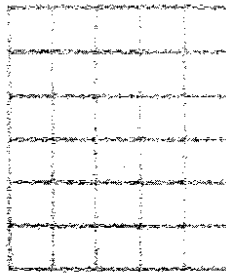
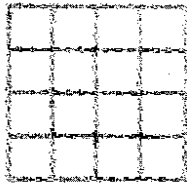
1.6



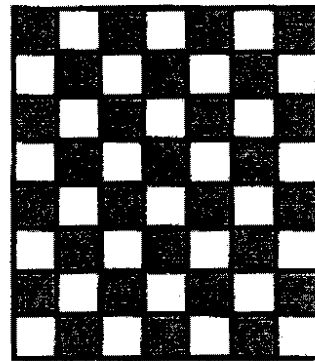
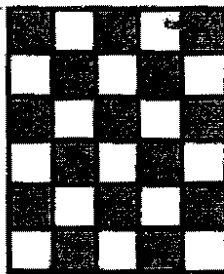
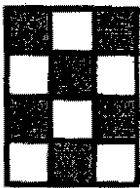
1.7



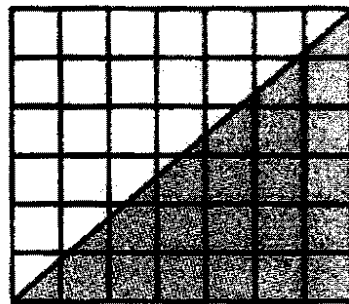
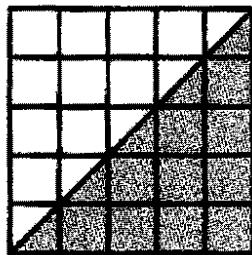
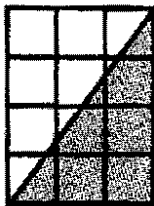
1.8



1.9



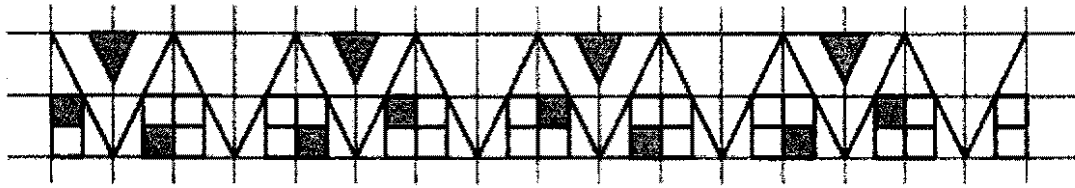
1.10



1.7 to 1.10

(Serra 1997: 47)

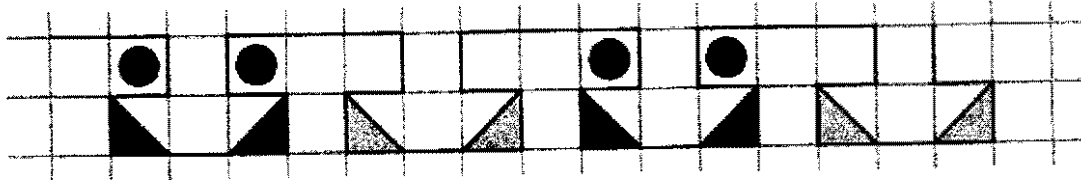
1.11



1.11

(Serra 1997: 47)

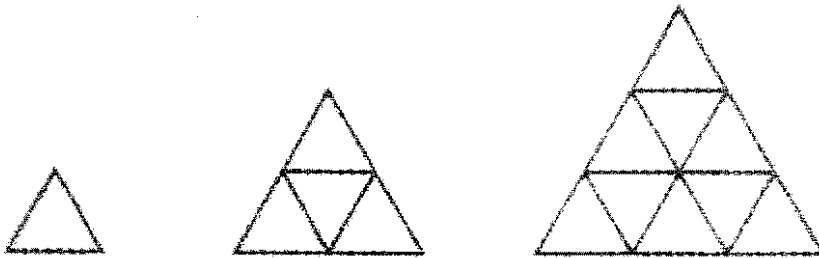
1.12



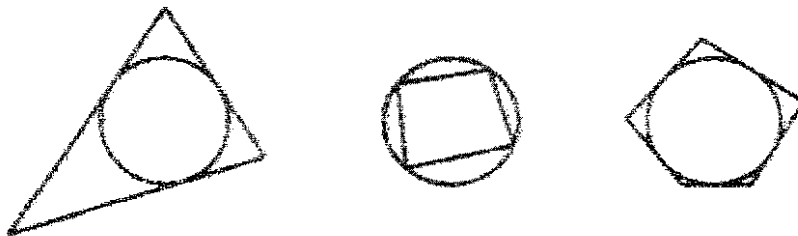
1.12

(Serra 1997: 48)

1.13



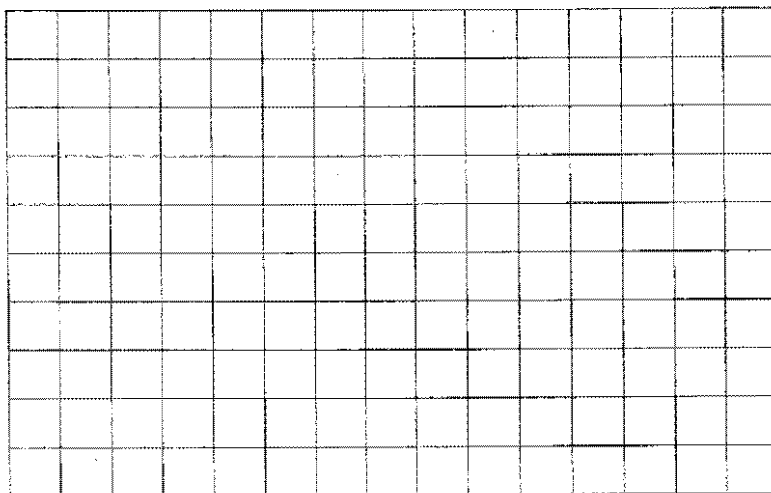
1.14



1.13 to 1.14 (Serra 1997: 62)

2.

Now it is your turn. Create a repeating pattern of your own in the grid provided.



Worksheet 1.4 Patterns

1. Establish patterns making use of the following given shapes or numbers:

1.1 \square

1.2 Δ

1.3 1

1.4 0

1.5 -3

1.6 x

1.7 y^2

2. Continue the following patterns:

2.1 1; 2;

2.2 7; 11;

2.3 0; 5;

2.4 3; -1;

2.5 $\frac{1}{2}$; 2;

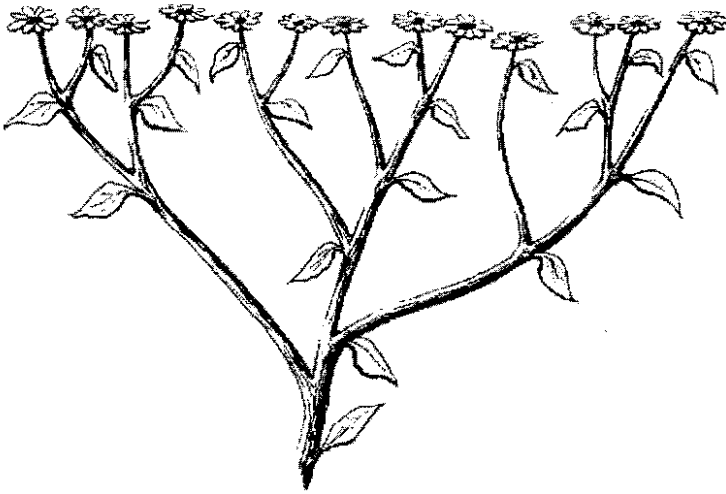
2.6 x ; $2x$;

2.7 y ; y^2 ;

XII

3. Study the following picture and tables carefully and discuss what sort of patterns you can see.

3.1



(Jacobs 1970: 76)

3.2

	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_0	ρ_0	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	ρ_0	μ_2	μ_3	μ_1
ρ_2	ρ_2	ρ_0	ρ_1	μ_3	μ_1	μ_2
μ_1	μ_1	μ_2	μ_3	ρ_0	ρ_2	ρ_1
μ_2	μ_2	μ_3	μ_1	ρ_1	ρ_0	ρ_2
μ_3	μ_3	μ_1	μ_2	ρ_2	ρ_1	ρ_0

(Fraleigh 1977: 40)

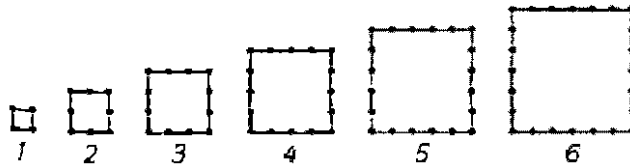
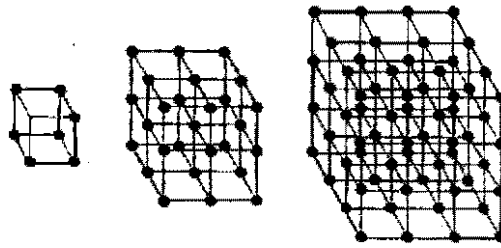
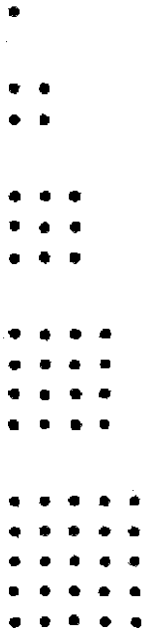
3.3

	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	δ_2	δ_1	μ_3	μ_2
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	μ_2	μ_1	δ_3	δ_1
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	δ_1	δ_2	μ_2	μ_1
μ_1	μ_1	μ_2	μ_3	δ_1	ρ_0	ρ_2	ρ_1	ρ_3
μ_2	μ_2	μ_3	μ_1	δ_2	ρ_2	ρ_0	ρ_3	ρ_1
δ_1	δ_1	δ_2	δ_3	μ_1	ρ_3	ρ_1	ρ_0	ρ_2
δ_2	δ_2	δ_1	μ_1	μ_2	ρ_1	ρ_3	ρ_2	ρ_0

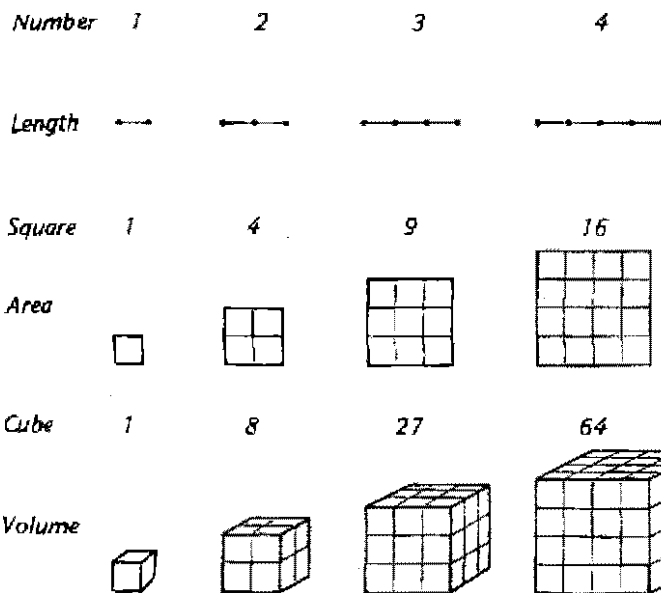
(Fraleigh 1977: 41)

Worksheet 2.1

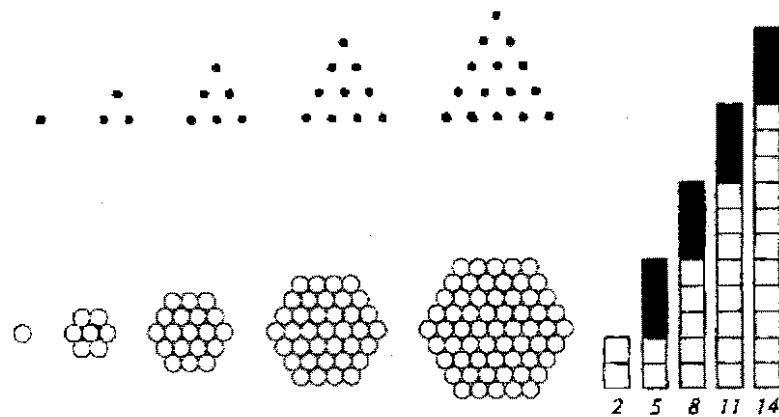
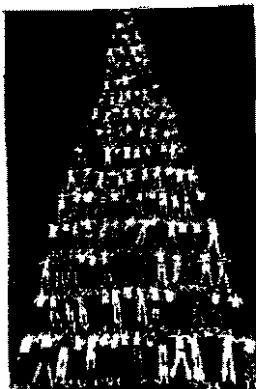
Sequences and Series



(Jacobs 1994: 62, 85 & 92)



(Jacobs 1994: 63 & 93)



(Jacobs 1994: 61, 65, 87 & 94)

Sequences

1.

Take a fairly large sheet of notebook paper and fold it in half, making a sharp crease. Turn the paper and fold it in half again at right angles to the first fold, again making a sharp crease.

You have now folded the paper twice. If the paper has been folded seven times, how many thicknesses of paper are in the wad produced?

To find out, we can make a list of the number of thicknesses of each wad starting with the first one.

No. of folds	:	1	2	3	4	5	6	7
No. of thicknesses	:	2	4	8	—	—	—	—

2.

Toss a coin. If you get heads write a 1 beneath the number of the throw and if you get tails write a 0 under the number of the throw as follows:-

No. of throw	:	1	2	3	4	5	6
Result (heads or tails)	:	0	0	1	?	?	?

Draw up your own chart in this way. After tossing the coin three times, can you guess what your fourth, fifth and sixth result will be?

3.

There is a legend that the King of Persia offered the inventor of the game of chess anything he wanted as a reward. What the inventor requested did not seem like much. He asked that one grain of wheat be placed on the first square, two grains on the second square, four grains on the third and so on, each square having twice the number of grains as the square before. The King thought that this was a reasonable request so he sent a servant for a sack of wheat. Can you guess what happened?



(Jacobs 1970: 54)

Let's begin with a small chessboard having only four squares. We can write the number of grains like this :-

1	2
4	8

1	2
$2 \cdot 2$	$2 \cdot 2 \cdot 2$

2^0	2^1
2^2	2^3

Doing the same with a slightly larger board we get :-

or

?

3.1

The number of grains on the last square of the 4-square board is 2^3 . The number of grains on the last square of the 9-square board is ? .

3.2

What do you think would be the number of grains on the last square of a 16-square board ?

3.3

What do you think would be the number of grains on the last square of a 64-square board ?

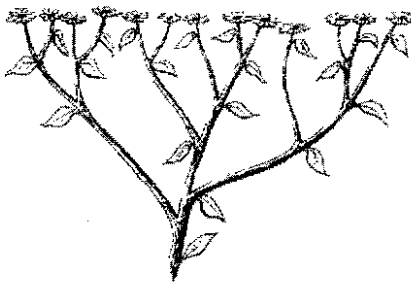
4.

Suppose that 27 members of a matric class are present at school on the first day and 30 on the second day, 23 on the third day and 29 on the fifth day. Have you any way of knowing how many pupils were present on the fourth day? Explain your answer.

Day	1	2	3	4	5
No. of pupils	27	30	23	?	29

5.

Any ordered list such as 2; 4; 6; is called a sequence. A Fibonacci sequence, named after Leonardo Fibonacci, is illustrated by the arrangement of leaves and flowers on the plant below. Try to detect the pattern and then write down the first 10 terms of the sequence.



(Jacobs 1970: 76)



Leonardo Fibonacci

(Jacobs 1970: 72)



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(Serre 1997: 43)

The numbers that solve this problem form the sequence

1 1 2 3 5 8 13 . . . ,

which is called the **Fibonacci sequence**.

The first two terms of the sequence are 1 and each succeeding term is the sum of the previous pair of terms.

THE greatest mathematician of the Middle Ages was Leonardo Fibonacci, born in Pisa, Italy. The construction of the famous Leaning Tower of that city was begun during his lifetime, but was not completed for nearly two centuries. In 1202, Fibonacci wrote a book on arithmetic and algebra in which he proposed the following problem:

A pair of rabbits one month old are too young to produce more rabbits, but suppose that in their second month and every month thereafter they produce a new pair. If each new pair of rabbits does the same, and none of the rabbits die, how many pairs of

$$1 \quad 1$$

$$1 + 1 = 2$$

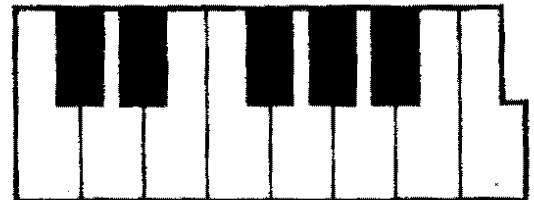
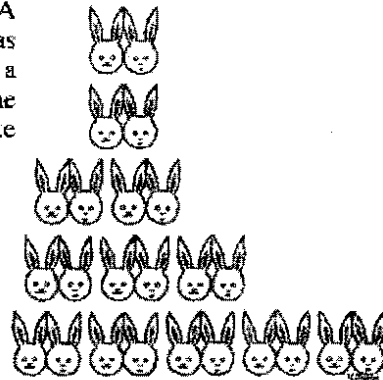
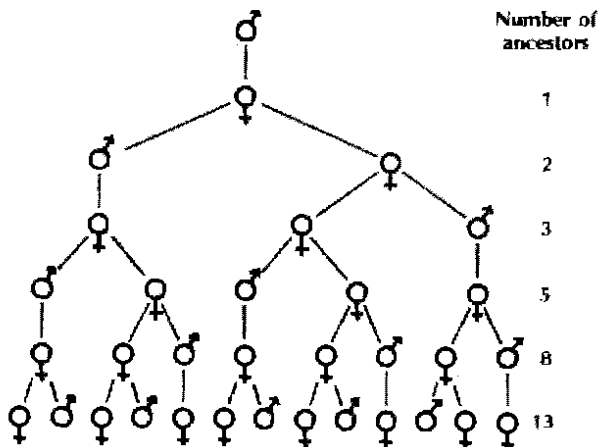
$$1 + 2 = 3$$

$$2 + 3 = 5$$

$$3 + 5 = 8$$

$$5 + 8 = 13$$

The Fibonacci number sequence appears in such unrelated topics as the family tree of a male bee and the keyboard of a piano! A male bee has only one parent, his mother, while a female bee has both father and mother. The family tree of a male bee has a strange pattern as a result. If each male is represented by the symbol ♂ and each female by the symbol ♀, the tree looks like this:



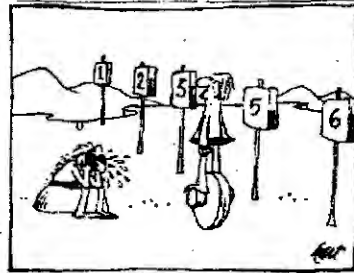
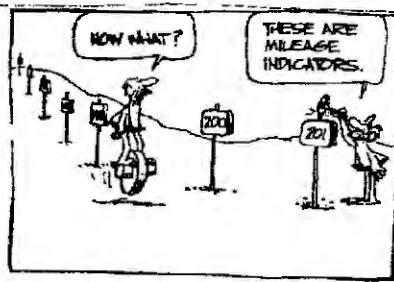
Another area of biology in which Fibonacci numbers appear is plant growth. The numbers of petals of many flowers are Fibonacci numbers. Not every flower is identical, but here are some typical values.*

3 petals	Lilies and irises
5 petals	Buttercups, larkspurs and columbines
8 petals	Some delphiniums
13 petals	Corn marigolds
21 petals	Some asters
34, 55, and 89 petals	Daisies

(Jacobs 1970: 72, 73, 74 & 76)

Worksheet 2.2

Sequences and Series

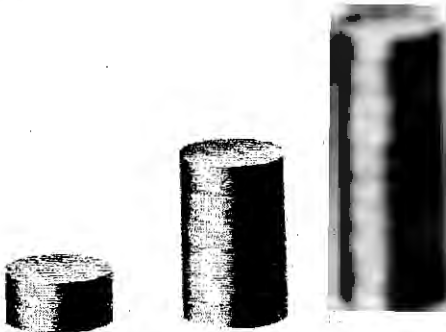


(Jacobs 1994: 60)

1.

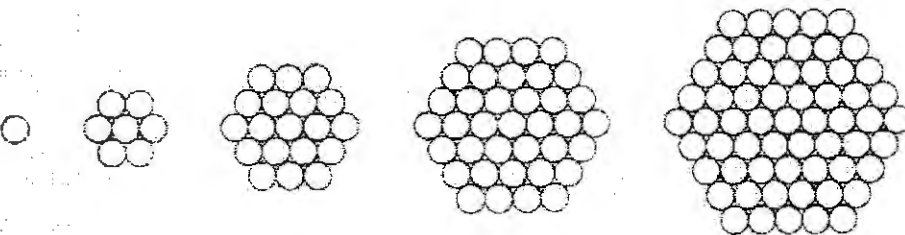
Write down the first six terms (numbers) of the number sequences suggested by the following figures:

1.1



(Jacobs 1970: 44)

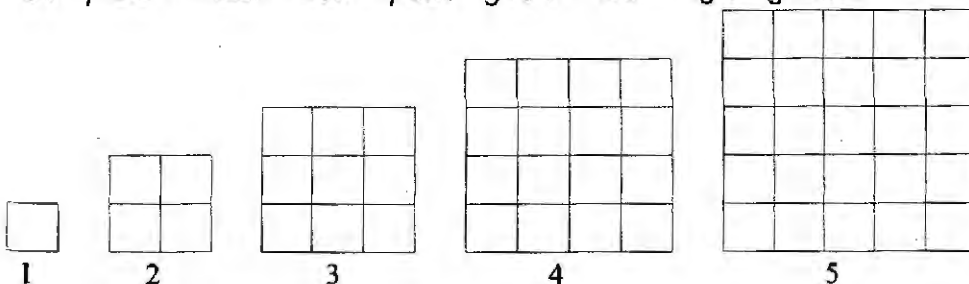
1.2



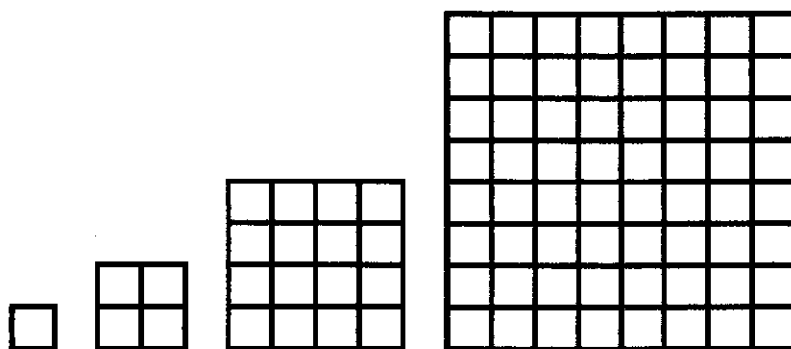
(Jacobs 1994: 94)

2.

Complete the tables corresponding to the following diagrams:



Number of term	1	2	3	4	5	6
Number of squares						



Number of term	1	2	3	4		
Number of squares						

3. T_1, T_2, T_3, T_4, T_5 and T_6 represent the 1st, 2nd, 3rd, 4th, 5th and 6th terms of a sequence. Try to fill in the missing spaces for the following sequences:

	T_1	T_2	T_3	T_4	T_5	T_6
6.1	3;	6;	9;			
6.2	4;	0;				-16;
6.3	-1;	-4;		-16;		
6.4	1;	2;				
6.5	1;	2;				
6.6	2;	4;				
6.7	2;	4;				
6.8	1;	4;				
6.9	1;	4;				
6.10	6;	3;				-9;
6.11	1;	0;	1;			
6.12	1;	20;	3;	7;		
6.13	1;	2^2 ;	3^3 ;	4^4 ;		
6.14	0;	1;	-7;		3;	
6.15	$\frac{1}{2}$;	$-\frac{1}{4}$;	$\frac{1}{8}$;			
6.16	1;	1;	2;			
6.17	1;	2^9 ;	3^4 ;			
6.18	2;	2;	4;			
6.19	m ;	$m + 1$;			$m + 4$;	
6.20	$\frac{q}{1}$;	$-\frac{q}{3}$;	$\frac{q}{5}$;			
6.21	2;	5;	9;			
6.22	44;	28;	20;			
6.23	2;	10;	30;	68;		
6.24	$\frac{2}{3}x$;	$-\frac{4x^3}{9y}$;	$\frac{8x^5}{27y^2}$;			

XIX

4 Fill in the missing spaces for the following sequences.

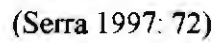
4.1.1 O; T; T; F; F; S; ; ;

4.1.2 2; 3; 2; 6; 3; 12; 7;

4.1.3 6; 8; 5; 10; 3; 11; 0;

4.2 Now make up a similar one of your own. Exchange your example with a person next to you and try to find out each other's pattern.

1.1

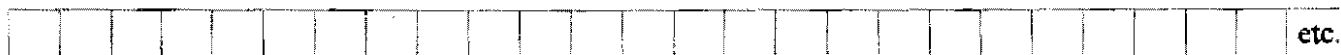
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(Serra 1997: 72)

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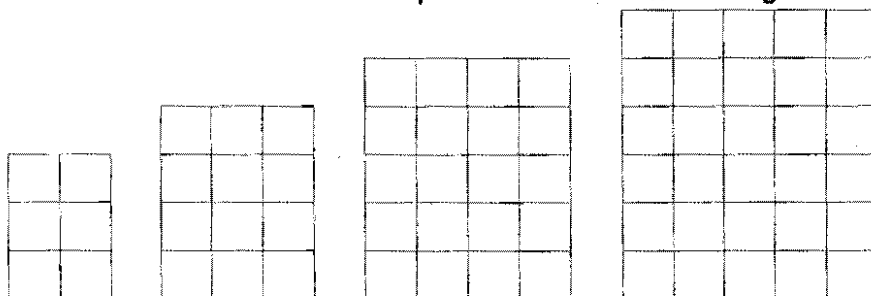
(Jacobs 1970: 62)

[illegible]

[illegible]

2.

What would the number of squares in the 200th rectangle be?

[illegible]

Worksheet 2.4

Sequences

1. Consider the sequence

2; 4; 6; _____; _____; _____; _____; _____; _____; _____;
 $\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow$
 $T_1 \quad T_2 \quad T_3 \quad T_4 \quad \quad \quad T_{10} \quad \quad \quad T_n \quad \quad \quad T_{100} \quad \quad \quad T_{642}$

We say:

$T_1 = 2$ which means that the first term is 2 or 2 occupies the first position in the sequence.

$T_2 = 4$ which means that the second term is 4 or 4 occupies the second position in the sequence.

$T_3 = 6$ which means that the third term is 6 or 6 occupies the third position in the sequence.

- 1.1 What would T_4 be?
 1.2 What would T_{10} be?
 1.3 What would T_n be?
 1.4 Use 1.3 to find T_{100} and T_{642} .

2. Find the values of T_4 , T_n , T_{100} in each of the following sequences:

					T_4	T_n	T_{100}
2.1	1;	2;	3;			
2.2	3;	6;	9;			
2.3	7;	14;	21;			
2.4	6;	0;	-6;			
2.5	1;	3;	5;			
2.6	10;	7;	4;			
2.7	1;	4;	9;			
2.8	2;	4;	8;			
2.9	3;	9;	27;			
2.10	1;	8;	27;			

3. If we are given T_n , then we can find the terms of a sequence. For example,

If $T_n = 3n - 1$,

then $T_1 =$

$T_2 =$

$T_3 =$

$T_{10} =$

$T_{100} =$

4. Find T_1 , T_2 , T_3 , T_{10} , and T_{100} , of the sequences for which

4.1 $T_n = 7n$

4.2 $T_n = -n + 1$

4.3 $T_n = n^2$

4.4 $T_n = 3n - 1$

4.5 $T_n = 2^n$

4.6 $T_n = 7$

4.7 $T_n = 3^n - 1$

4.8 $T_n = \frac{2n}{2n+1}$

4.9 $T_n = (-1)^n n^3$

4.10 $T_n = n^2 - 2n$

WORKSHEET 2.5 SEQUENCES

1. Write down the first four terms of the sequence with the following general terms:

1.1 $T_k = k - 3$

1.2 $T_k = k^2 + 1$

1.3 $T_k = 2^k$

1.4 $T_k = 3 \cdot 2^{k+1}$

1.5 $T_k = (-1)^k$

1.6 $T_k = -2(-1)^{k+1}$

2. Find the indicated terms of the sequences with general terms stated below:

2.1 T_1, T_4, T_{100} if $T_k = 2k - 1$

2.2 T_2, T_6, T_{200} if $T_k = 2 \cdot 3^{k-1}$

2.3 T_1, T_7, T_{126} if $T_k = 2k^2 + 1$

3. Determine a formula for the k th term of each of the following sequences:

3.1 2; 4; 6; 8; - - - - -

3.2 -3; -6; -9; -12; - - - - -

3.3 1; 3; 5; 7; - - - - -

3.4 1; 4; 7; 10; - - - - -

3.5 1; 4; 9; 16; - - - - -

3.6 -2; -2; -2; -2; - - - - -

3.7 1; 8; 27; 64; - - - - -

3.8 1; -1; 1; -1; - - - - -

3.9 $\frac{1}{2}; \frac{2}{3}; \frac{3}{4}; \frac{4}{5}; - - - - -$

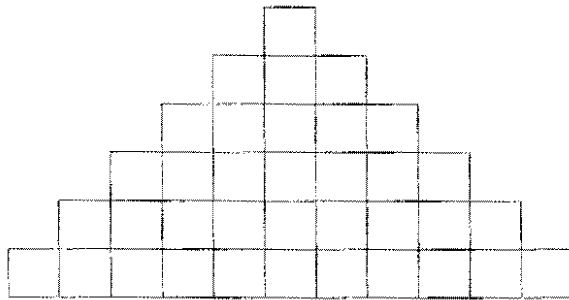
3.10 -3; 9; -27; 81 - - - - -

WORKSHEET 3.1 ARITHMETIC SEQUENCES

1. Consider the following examples and non-examples of arithmetic sequences.

Examples

E1



E2

3; 6; 9;

E3

The amount of money which could be won in a radio jackpot, which begins with R200-00 and, until it is won, increases by R50-00 per hour.

E4

1; 7; 13; 19;

E5

6; 0; -6;

E6

9; 9; 9;

E7

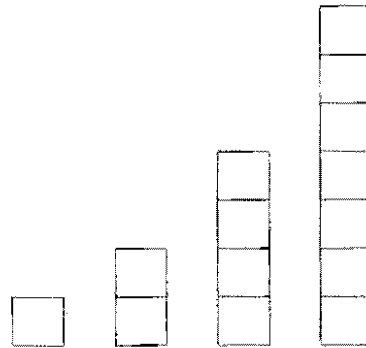
The amount of money John saves if he starts off with R70-00 and adds R10-00 per weekend (without adding or subtracting any other money to the original amount).

E8

23,1; 23,3; 23,5;

Non-Examples

N1



(Cangelosi 1996: 90)

N2

2; 4; 8;

N3

The ages of all the people in your class at school, listed in terms of the alphabetical order of their names.

N4

1; 7; 19;

N5

6; -6; 6;

N6

9; -9; 9;

N7

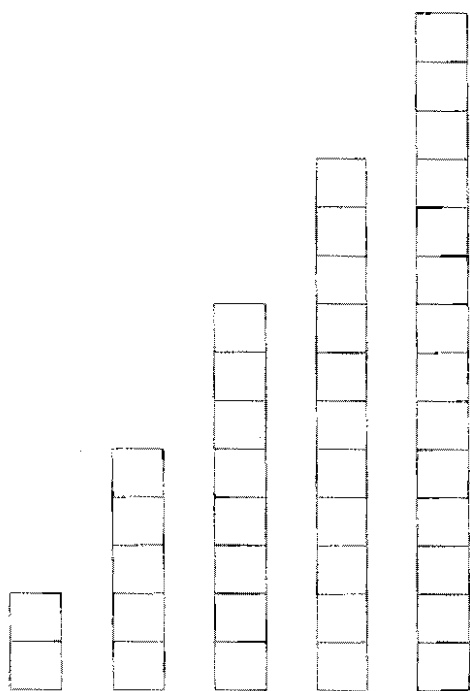
The monthly savings of Jane if she deposits R500-00 and leaves it there to accumulate 50% interest compounded annually.

N8

23,1; 23,01; 23,001;

E9

XXVI
N9



E10

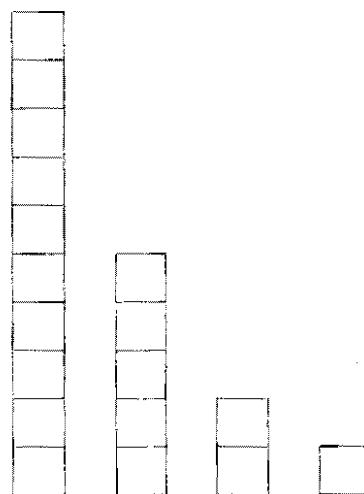
-6; -6,1; -6,2;

E11

7; -3; -13;

E12

$x + 2y$; $3x + 3y$; $5x + 4y$;



N10

6,1; 6,12; 6,123;

N11

7; -3; -20;

N12

$x + 2y$; $3x + 3y$; $6x + 4y$;

(Cangelosi 1996: 90)

2.

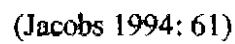
2.1 How are the examples alike?

2.2 Are the non-examples alike?

2.3 How do the examples and the non-examples differ?

2.4 How do the examples differ from each other?

- 3.



- 3.2 Complete the following table:

T_1	T_2	T_3	T_4	T_5	T_n	T_{100}
2	5	8				
2+	2+	5+				
2+	2+	2+				
2+ \cdot	2+ \cdot	2+ \cdot				

4. Copy and complete each table. Find the differences between consecutive values. What do you notice?

4.1

n	1	2	3	4	5	6	7	8
$n - 5$	-4	-3						

4.2

n	1	2	3	4	5	6	7	8
$2n + 1$	3	5						

4.3

n	1	2	3	4	5	6	7	8
$3n - 2$	1	4						

4.4

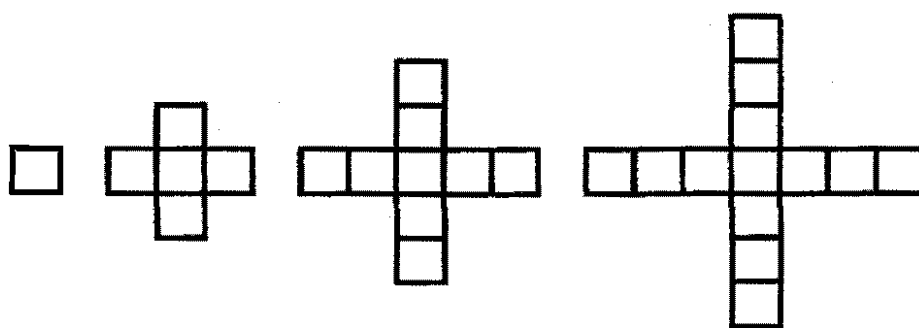
n	1	2	3	4	5	6	7	8
$5n + 7$	12	17						

(Serra 1997: 53)

5. In question 4 did you spot a pattern? Describe it and then complete the following table:

n	1	2	3	4	5	90
$7n + 20$	$7(1) + 20 =$					

6. Complete the tables for the following example:



(Serra 1997: 53)

Squares in a cross array.

Cross	1	2	3	4	5	6	n	200
Number of squares	1	5								

WORKSHEET 3.3 ARITHMETIC SEQUENCES

1. Consider the arithmetic sequence

3; 10; 17;

- 1.1 What are the 4th, 5th and 6th terms?
- 1.2 What is the 10th term?
- 1.3 What is the 20th term?
- 1.4 How would you predict T_{100} without performing repeated addition?
- 1.5 Could you predict T_n ?

2. Consider the arithmetic sequence

7; 2; -3;

- 2.1 What are T_4 , T_5 and T_6 ?
- 2.2 Find T_{10} .
- 2.3 Find T_{20} .
- 2.4 How would you predict T_{100} ?
- 2.5 Could you predict T_n ?

3. Suppose that a is the first term and d is the common difference of an arithmetic sequence.

1st term

a

2nd term

a	d
-----	-----

3rd term

a	d	d
-----	-----	-----

4th term

a	d	d	d
-----	-----	-----	-----

5th term

a	d	d	d	d
-----	-----	-----	-----	-----

(Jacobs 1994: 65)

- 3.1 List T_1 , T_2 , T_3 , T_4 , T_5 and T_6 .
- 3.2 If k is any natural number, find an expression for T_k .
- 3.3 What is T_{1000} ?

WORKSHEET 3.4 ARITHMETIC SEQUENCES

Write down the formula you have derived in worksheet 3.2 for the general term T_k of an arithmetic sequence with first term a and common difference d :

1. List the first three terms of an arithmetic sequence with

1.1 $a = 2 ; d = 3.$

1.2 $a = 3 ; d = 5.$

1.3 $a = 0,2 ; d = -1.$

1.4 $a = -\frac{3}{4} ; d = \frac{1}{2}.$

1.5 $a = 2x + y ; d = -x - y.$

2. Determine T_{20} and T_{97} for the following arithmetic sequences:

2.1 $2; 7; 12; \dots\dots\dots$

2.2 $3; \frac{1}{2}; -2; \dots\dots\dots$

2.3 $2 + 3x; 5 + 5x; 8 + 7x; \dots\dots\dots$

2.4 $a = -2; d = 7$

2.5 $a = \frac{1}{3}; d = -\frac{2}{3}$

3. Given the general terms of the following arithmetic sequences, find a, d and T_{27} .

3.1 $T_k = 3k + 2$

3.2 $T_k = 7 - 4k$

4. Find which term of the sequence

4.1 $2; 9; 16; \dots\dots\dots$ equals 149.

4.2 $-2m; -6m; -10m; \dots\dots\dots$ equals $-170m$

5. Find the arithmetic sequence with

5.1 First term 3 and 21st term 87.

5.2 First term -10 and 50th term 15.

5.3 $T_5 = 19$ and $T_{11} = 37$.

5.4 $T_4 = -15$ and $T_{12} = -63$.

6. Given the arithmetic sequence

$-4\frac{1}{4}; -3\frac{1}{2}; -2\frac{3}{4}; \dots\dots\dots$

6.1 Which term in the sequence is 13?

6.2 Calculate T_{13} .

7. If $3y - 1$; $-2y + 3$ and $2y - 1$ are the first three terms of an arithmetic sequence

7.1 Calculate y .

7.2 Determine the first 3 terms of the sequence.

7.3 Determine a formula for the k th term.

7.4 Which term in the sequence is -61 ?

7.5 Calculate the 19th term.

WORKSHEET 3.5 ARITHMETIC SEQUENCES

1.

- 1.1 Insert a number x between 3 and 11 so that 3; x ; 11 forms an arithmetic sequence.

The number x here is called an *arithmetic mean*.

- 1.2 How would you define an arithmetic mean?

- 1.3 What is the arithmetic mean of a and b ?

2. Find an arithmetic mean between the following pairs of terms:

2.1 9; 21

2.2 -11 ; 33

2.3 $-4\frac{1}{2}$; $\frac{1}{2}$

2.4 $x - 6$; $2x + 1$

3. Find 3 numbers T_2 , T_3 and T_4 such that 2; T_2 ; T_3 ; T_4 ; 10 forms an arithmetic sequence.

T_2 , T_3 and T_4 here are called the three arithmetic means between 2 and 10.

4. Find 6 arithmetic means between 11 and 106.

5. Find 9 arithmetic means between 20 and -61 .

6. Find 7 arithmetic means between $2x + 2$ and $23x - 5$.

1. Consider the following examples and non-examples of geometric sequences.

Examples

Non-Examples

E1

N1



(Jacobs 1970: 54)

E2 1; 3; 9;



(Jacobs 1970: 76)

N2 1; 3; 6;

E3 The list of the number of thicknesses of paper in a wad produced by folding a sheet of paper in half, where each time the crease is at right angles to the previous fold.

N3 The list of results obtained by throwing a coin repeatedly and writing down 1 for heads and 0 for tails.

E4 3; 6; 12;

N4 3; 6; 9;

E5

The figure shows a 5x5 grid of dots. The dots are arranged in a pattern that suggests a sparse matrix, with some dots missing from the main diagonal and off-diagonal elements.

N5

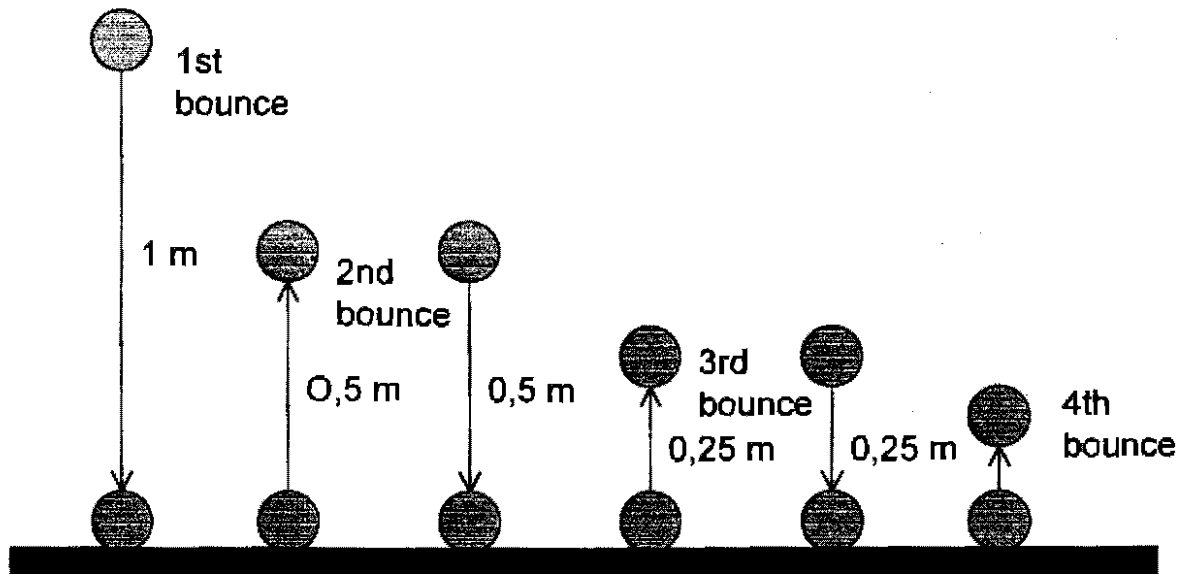
Circumstance	Percentage
1. Yes, always	0%
2. Yes, most of the time	10%
3. Yes, some of the time	30%
4. Yes, never	10%
5. No, never	10%
6. No, some of the time	30%
7. No, most of the time	10%
8. No, always	10%
9. Don't know	10%
10. Yes, always	10%
11. Yes, most of the time	10%

E6 1; -1; 1; -1;

N6 1;-1;-1;1;

E7 The sequence of the heights of the bounces of a ball dropped from a height of 1 metre and bouncing up exactly one half the distance it has just come down.

N7 The list of the number of pupils of a class of 30 (taken in alphabetical order), who were present each day of the second term this year.



E8 $4; 1; \frac{1}{4}; \dots$

N8 $4; 1; -4; \dots$

E9 $\frac{2}{9}; \frac{1}{9}; \frac{1}{18}; \dots$

N9 $\frac{2}{9}; \frac{4}{9}; \frac{6}{9}; \dots$

E10 $-5; -10; -20; -40; \dots$

N10 $-5; 10; 20; -40; \dots$

E11 $0,7; 0,07; 0,007; \dots$

N11 $0,7; 0,77; 0,777; \dots$

E12 $3x; 3x^2y^2; 3x^3y^4; \dots$

N12 $3xy; 6xy; 9xy; \dots$

2.

2.1 How are the examples alike?

2.2 How are the non-examples alike?

2.3 How do the examples and non-examples differ?

2.4 How do the examples differ from each other?

WORKSHEET 4.2 GEOMETRIC SEQUENCES

1. Give three examples of sequences of numbers that are:
 - 1.1 Geometric.

- 1.2 Non-geometric.

2. How would you define a geometric sequence?

3. In his "Poor Richard's Almanack" for the year 1751, Benjamin Franklin calculated the number of person's ancestors 30 generations back, or approximately 1000 years ago. Franklin's list begins like this:

A present Man's Father and Mother were _____
 His Grandfathers and Grandmothers _____
 His Great Grandfathers and Great Grandmothers _____
 and, supposing no Intermarriages among Relations,
 the next Predecessors will be _____

The number sequence of ancestors,

_____, _____, _____, _____,

- 4.

- 4.1 Examine the United States currency notes shown on the next page and then list all of the geometric sequences containing at least three terms that you can find in the denominations of these notes.

- 4.2 One of the sequences you probably found is

5 10 20.

Is the tenth term of this sequence also the denomination of a currency note?

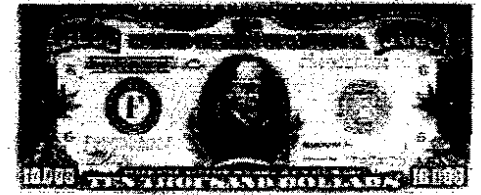
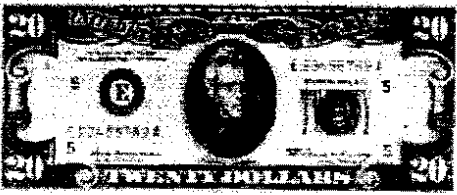
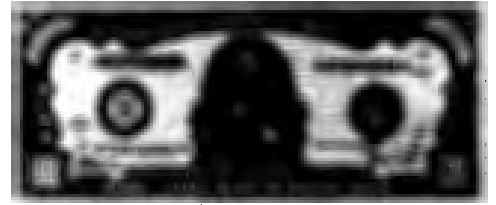
- 4.3 Another way to represent the terms of this sequence is:

1st term	2nd term	3rd term	4th term	5th term
5	$5 \cdot 2$	$5 \cdot 2 \cdot 2$	$5 \cdot 2 \cdot 2 \cdot 2$	$5 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Rewrite the 3rd, 4th and 5th terms, using exponents. (An exponent is a number written at the upper right of another number to indicate how many times that number occurs as a factor. For example, $5 \cdot 2 \cdot 2 \cdot 2$ can be written as $5 \cdot 2^3$.)

- 4.4 Notice the relationship between the exponent and the number of the term. Write the 10th term of this sequence, using an exponent.

- 4.5 Write the 100th term, using an exponent.



Courtesy of the U.S. Department of the Treasury

(Jacobs 1994: 71)

7

[illegible][illegible]

1.1 What type of sequence does this represent?
Explain your answer

T_1	T_2	T_3	T_4	T_5
1	5	25		
$1 \times \underline{\quad}$	$5 \times \underline{\quad}$	$25 \times \underline{\quad}$		
$1 \times \underline{\quad}$	1×5	$1 \times \underline{\quad}$		
$1 \times 5^?$	$1 \times \underline{\quad}$	$1 \times \underline{\quad}$		

2.1	n	1	2	3	4	5
	2^n					

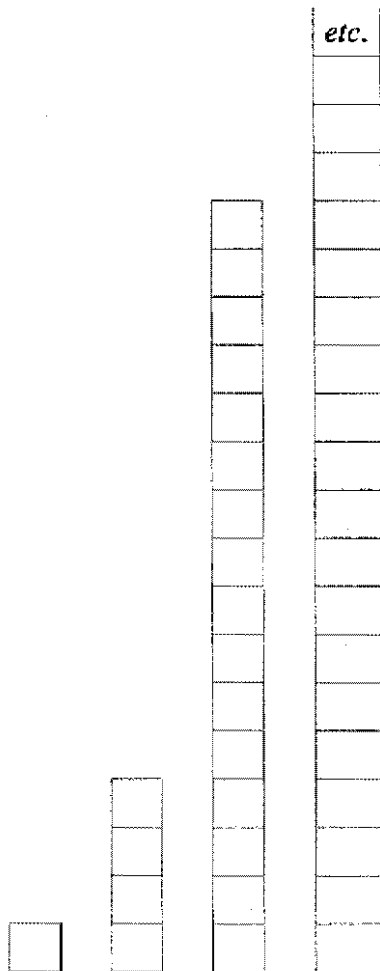
2.2	n	1	2	3	4	5
	$\frac{1}{3^n}$					

2.3	n	1	2	3	4	5
	2^{-n}					

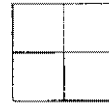
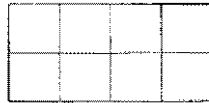
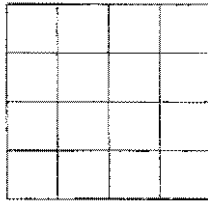
2.4	n	1	2	3	4	5
	$4\left(-\frac{2}{3}\right)^n$					

k	1	2	3	4	5	90
$2 \cdot (-3)^n$	$2(-3)^1$						
$=$							

4. Complete the table for the following example:
- 4.1

[illegible]

4.2



Term number	1	2	3	4	5	6	n	100
Number of squares										

5. Consider the sequence 2; 6; 18;
- 5.1 Is it geometric? Why?
 - 5.2 What are the 4th, 5th and 6th terms?
 - 5.3 What is the 10th term?
 - 5.4 What is the 20th term?
 - 5.5 How would you predict T_{100} without performing repeated multiplication?
6. Consider the sequence 8; -4; 2; -1;
- 6.1 Is it geometric? Why?
 - 6.2 Find T_{10} .
 - 6.3 Find T_{20} .
 - 6.4 How would you predict T_{100} ?
7. Suppose that a is the first term and r is the common ratio of a geometric sequence.
- 7.1 List $T_1, T_2, T_3, T_4, T_5, T_6$.
 - 7.2 If k is any rational number, find an expression for T_k .
 - 7.3 What is T_{1000} ?

WORKSHEET 4.4 GEOMETRIC SEQUENCES

Write down the formula you have derived in worksheet 4.3 for the general term T_k of a geometric sequence with first term a and common ratio r .

Use this formula to help you answer the following questions:

1. List the first three terms of a geometric sequence with
 - 1.1 $a = 3$; $r = 2$
 - 1.2 $a = -2$; $r = -3$
 - 1.3 $a = 1$; $r = \frac{1}{2}$
 - 1.4 $a = \frac{1}{4}$; $r = -2$
 - 1.5 $a = 2x$; $r = \frac{1}{4}y$
2. Determine T_{30} and T_{106} for the following geometric sequences:
 - 2.1 1; 4; 8;
 - 2.2 -3; -15; -75;
 - 2.3 6; -12; 24;
 - 2.4 $\frac{2}{3}; \frac{4}{9}; \frac{8}{27}; \dots$
 - 2.5 $\frac{m}{n^2}; -\frac{m}{n}; m^3; \dots$
3. Given the general terms of the following geometric sequences, find a , r and T_{34} .
 - 3.1 $T_k = 6^k$
 - 3.2 $T_k = 4 \cdot 2^{k-1}$
 - 3.3 $T_k = 2 \cdot 5^{-k}$
4. Find which term of the sequence
 - 4.1 1; 4; 16; is 4096
 - 4.2 $3; -\frac{3}{2}; \frac{3}{4}; \dots$ is $\frac{-3}{128}$
 - 4.3 $\frac{a^2}{7}; 1; \frac{7}{a^2}; \dots$ is $\frac{343}{a^6}$
5. Find the geometric sequence with
 - 5.1 First term 3 and 10th term 1536.
 - 5.2 First term 5 and 5th term $\frac{5}{81}$.
 - 5.3 $T_3 = 36$ and $T_7 = 2916$.
 - 5.4 $T_6 = -\frac{3}{16}$ and $T_9 = \frac{3}{128}$
 - 5.5 $T_4 = \frac{mn^4 + n^4}{8}$ and $T_{12} = \frac{mn^{11} + n^{12}}{2^{11}}$
6. Given the sequence $\frac{3}{16}; \frac{3}{8}; \frac{3}{4}; \dots$
 - 6.1 Which term of the sequence is 36?
 - 6.2 Calculate T_{36} .
7. $y - 4$; $y + 2$ and $3y + 1$ are the 4th, 5th, and 6th terms respectively of a geometric sequence. Find two possible values for
 - 7.1 the common ratio.
 - 7.2 the first term.

WORKSHEET 4.5 GEOMETRIC SEQUENCES

1.

- 1.1 Insert a number x between 4 and 100 so that 4 ; x ; 100 forms a geometric sequence.

The positive answer here (unless otherwise specified) is called the **geometric mean**.

- 1.2 How would you define a geometric mean?

- 1.3 What is the geometric mean of a and b ?

- 1.4 Compare the arithmetic mean to the geometric mean of two terms a and b .

2. Find a geometric mean between the following pairs of terms.

- 2.1 5 ; 45

- 2.2 -6 ; -24

- 2.3 $-\frac{1}{3}$; -27

- 2.4 $\frac{5m^2}{n}$; $\frac{5m^4}{9n^3}$.

3. Find 3 numbers T_2 , T_3 and T_4 such that 3, T_2 , T_3 , T_4 , 768 forms a geometric sequence. T_2 , T_3 and T_4 are called three geometric means between 3 and 768.

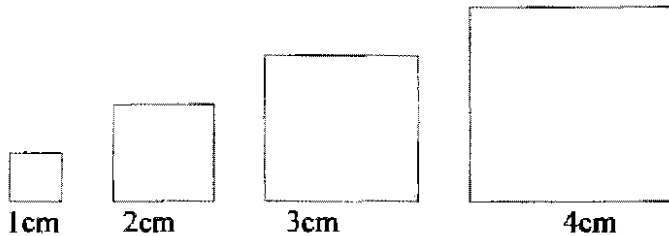
4. Insert five terms between 7 and 109375 so as to obtain a geometric sequence.

5. Insert four geometric means between $2x$ and $\frac{x^6}{486}$.

6. Insert five geometric means between ab and $\frac{64a}{b^5}$.

WORKSHEET 5.1 SERIES

1. Consider the following sequences of squares with sides of length 1cm, 2cm, 3cm and 4cm.



- 1.1 Suppose we were to construct squares of these sizes using a thin piece of wire. State how much wire would be needed for the

first square _____
 second square _____
 third square _____
 fourth square _____

- 1.2 If we consider the sequence of lengths of wire required to construct the square then

$$T_1 =$$

$$T_2 =$$

$$T_3 =$$

$$T_4 =$$

and the sequence could be written as

_____ ; _____ ; _____ ; _____ ;

- 1.3 Now, how much wire would be required for square 1 and square 2:?

$$\text{_____} + \text{_____} =$$

How much wire would be required for squares 1, 2 and 3:?

$$\text{_____} + \text{_____} + \text{_____} =$$

How much wire would be required for squares 1, 2, 3 and 4:?

$$\text{_____} + \text{_____} + \text{_____} + \text{_____} =$$

- 1.4 There are often times when we wish to add up the number of terms in a sequence as we saw in the above situation. If S_k stands for the sum of the first k terms then in the case of the squares above:

$$S_1 = \text{_____}$$

$$S_2 = \text{_____} + \text{_____} =$$

$$S_3 = \text{_____} + \text{_____} + \text{_____} =$$

$$S_4 = \text{_____} + \text{_____} + \text{_____} + \text{_____} =$$

2. If T_1, T_2, T_3, T_4, T_5 , and T_6 denote the first 6 terms of a sequence, then

$$S_1 =$$

$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

$$S_6 =$$

3. In the previous example, we wrote

$$S_4 = T_1 + T_2 + T_3 + T_4$$

Here we expressed S_4 as the sum of the first four terms of a sequence.

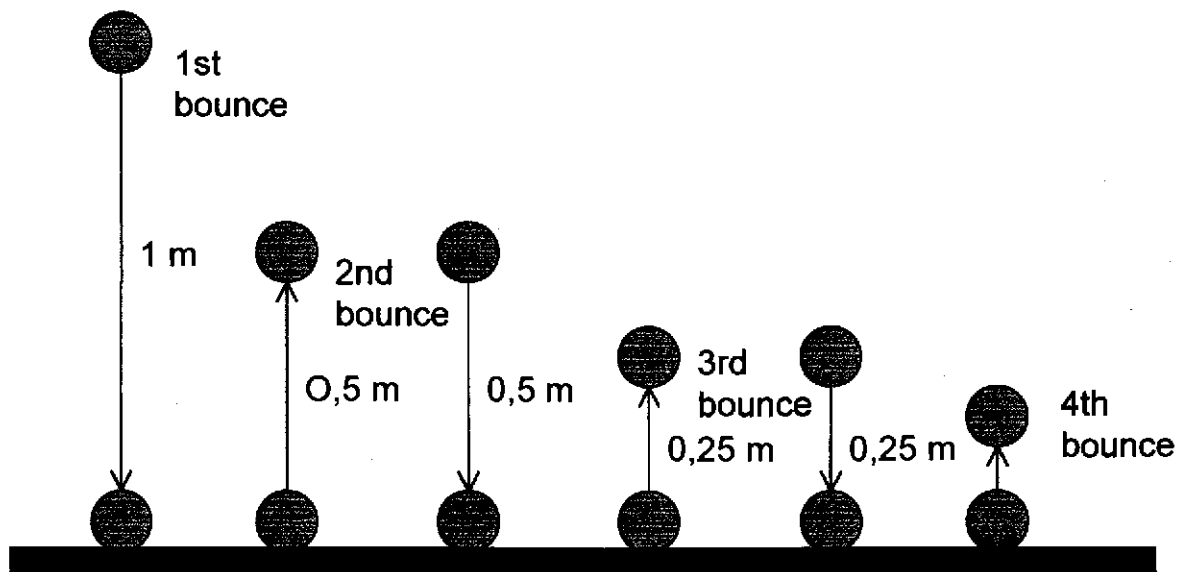
If we consider the sequence

$$a; a^2; a^3; a^4; a^5; a^6$$

What is S_6 ?

What is T_6 ?

4. Consider the sequence of the heights of the bounces of a ball dropped from a height of 1 metre and bouncing straight up and down, always bouncing up exactly one half the distance it has just come down.



- 4.1 What sort of sequence is represented in this situation?
- 4.2 What is the total distance travelled by the ball from the moment it was first dropped to the time it bounced on the ground for the third time?

4.3 How did you obtain your answer in 4.2?

From a sequence we may obtain a series by expressing the terms of a sequence as a sum.

Thus we define a series as follows:

A series is the indicated sum of the terms in a sequence.

For example, the sequence 3; 7; 11; 123
may be associated with the series $3 + 7 + 11 + \dots + 123$

5. In general, corresponding to the sequence $T_1, T_2, T_3, \dots, T_n$, we have the series where T_k is the general term of the sequence for $k \in \{1; 2; 3; \dots, n\}$.
If only n terms are added we have a finite series or partial sum and this is also called a series of n terms.

(Laridon 1996: 62)

6. The series $T_1 + T_2 + T_3 + \dots$ is an infinite series.
Give an example of an infinite series of numbers.

WORKSHEET 5.2 SERIES

1. Write down the corresponding series for each of the given sequences below:
 - 1.1 3; 6; 9; 12; 15; 18; 21.
 - 1.2 4; 9; 14;94.
 - 1.3 1; 7; 49;
 - 1.4 -5; 0; -5; 0;
 - 1.5 -11; -22; -33; - 99.
 - 1.6 $x; -x^2y; x^3y^2$
2.
 - 2.1 Write down the first six terms of an arithmetic sequence with first term 6 and common difference 7.
 - 2.2 Hence write down the corresponding series.
 - 2.3 Find the sum of the series in 2.2.
3.
 - 3.1 Write down the first seven terms of a geometric sequence with first term 2 and common ratio -3.
 - 3.2 Hence write down the corresponding series.
 - 3.3 Find the sum of the corresponding series.
4. If $T_k = 2 + (k - 1)3$, find:
 - 4.1 the first nine terms of the sequence.
 - 4.2 the corresponding series of 9 terms.
 - 4.3 S_9
5. If $T_k = 2 \cdot 3^{k-1}$, find.
 - 5.1 the first five terms of the sequence.
 - 5.2 the corresponding series of 5 terms.
 - 5.3 S_5
6. If $T_k = kx^{k+1}$ find
 - 6.1 all the terms of the sequence.
 - 6.2 the corresponding series.
7. If $T_k = (-1)^k$ find
 - 7.1 S_{10}
 - 7.2 S_{11}
 - 7.3 S_{1000}
 - 7.4 S_{2007}

8. If $S_{20} = 20$ and $S_{19} = 17$, what is T_{20} ?

9. If $S_n = 2n^2 + n$, find the values of

9.1 S_1

9.2 S_2

9.3 S_3

9.4 T_1

9.5 T_2

9.6 T_3

9.7 T_n

10. For a certain series, $S_n = 3^n - 1$.

10.1 Find the values of S_1, S_2, S_3 and S_4 .

10.2 Find the values of T_1, T_2, T_3 and T_4 .

10.3 State what type of sequence gives rise to this series.

WORKSHEET 5.3 SERIES

1. In example 4 of worksheet 5.2, we found that the first nine terms of the sequence with general term $T_k = 2 + (k-1)3$ are

2; 5; 8; 11; 14; 17; 20; 23; 26

with corresponding series

$$2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26$$

Instead of writing $S_9 = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26$

We have another useful form of notation, called sigma notation in which we make use of the Greek letter \sum (read sigma and the capital letter for S) to denote summation.

In this example, we would write

$$S_9 = \sum_{k=1}^9 [2 + (k-1)3]$$

because

$$\begin{aligned} & \sum_{k=1}^9 [2 + (k-1)3] \\ = & [2 + (1-1)3] + [2 + (2-1)3] + [2 + (3-1)3] + [2 + (4-1)3] + [2 + (5-1)3] \\ & + [2 + (6-1)3] + [2 + (7-1)3] + [2 + (8-1)3] + [2 + (9-1)3] \\ = & [2 + (0)3] + [2 + (1)3] + [2 + (2)3] + [2 + (3)3] + [2 + (4)3] \\ & + [2 + (5)3] + [2 + (6)3] + [2 + (7)3] + [2 + (8)3] \\ = & 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 \end{aligned}$$

- 1.1 Why do you think the letter \sum is used?
- 1.2 What do the $k=1$ and the 9 above and below the \sum sign mean?
- 1.3 What does $2 + (k-1)3$ represent?
- 1.4 What do you think are the advantages of this notation?

When we are asked to expand $\sum_{k=1}^9 [2 + (k-1)3]$ we write

$$\sum_{k=1}^9 [2 + (k-1)3] = 2 + 5 + 8 + \dots + 26$$

If there are many terms then we may write down the first three terms to show the pattern and then the last one to show where to stop as has been done above.

However, if we asked to calculate $\sum_{k=1}^9 [2 + (k-1)3]$ then we add up all the terms

and in this case we obtain $\sum_{k=1}^9 [2 + (k-1)3] = 126$.

2. Now consider the series $\sum_{k=2}^7 2 \cdot 3^{k-1}$.

2.1 What do the $k = 2$ and 7 represent?

2.2 What does $2 \cdot 3^{k-1}$ represent?

2.3 Expand $\sum_{k=2}^7 2 \cdot 3^{k-1}$.

2.4 Evaluate $\sum_{k=2}^7 2 \cdot 3^{k-1}$.

3. Consider the series $\sum_{k=1}^{\infty} k^2$

3.1 What do k and ∞ represent?

3.2 What does k^2 represent?

3.3 Expand $\sum_{k=1}^{\infty} k^2$

3.4 Can you evaluate $\sum_{k=1}^{\infty} k^2$?

WORKSHEET 5.4 SERIES

- 1.1 What is the general term of the sequence
2; 4; 6;?
- 1.2 Can you express the series $2 + 4 + 6 + \dots$ in sigma notation?
- 2.1 What is the general term of the sequence
3; -1; -5;?
- 2.2 Express the series $3 - 1 - 5 - \dots$ in sigma notation.
- 3.1 What is the general term of the sequence
3; 6; 12;?
- 3.2 Express the series
 $3 + 6 + 12 + \dots$ in sigma notation.

4. Express in sigma notation.

- 4.1 $1 + 2 + 3 + \dots + 24$.
- 4.2 $3 + 7 + 11 + \dots$
- 4.3 $-7 - 7 - 7 - \dots$
- 4.4 $-2^2 - 3^2 - 4^2 - 5^2 - \dots - n^2$.
- 4.5 $1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{1024}$.

5. Expand the following series

5.1 $\sum_{k=1}^6 (2k - 1)$

5.2 $\sum_{k=3}^{20} 3 \cdot 2^k$

5.3 $\sum_{m=1}^{\infty} (m^2 - 1)$

5.4 $\sum_{s=1}^n (3s - 2)$

6. Expand and evaluate the following series:

6.1 $\sum_{k=1}^{10} (2k + 3)$

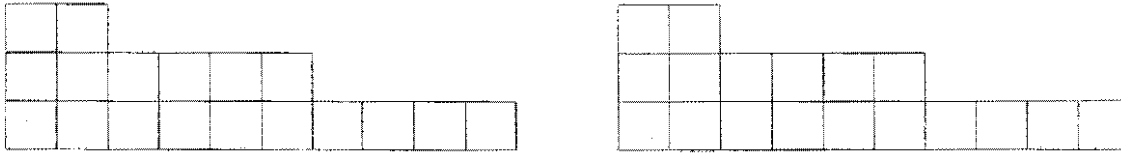
6.2 $\sum_{n=3}^{12} 9$

6.3 $\sum_{m=1}^7 -m^2 + 1$

6.4 $\sum_{k=2}^{20} \frac{1}{4}(-1)^k$

WORKSHEET 6.1 ARITHMETIC SERIES

1. Consider the series
 $2 + 6 + 10$
 Illustrated twice below:



- 1.1 Cut out both the block patterns and combine them to create a rectangle.
 - 1.2 What is the width of the rectangle?
 - 1.3 What is the length of the rectangle?
 - 1.4 What numerical characteristic of the series is equal in value to the width of the rectangle?
 - 1.5 What numerical characteristic of the series is equal in value to the length of the rectangle?
 - 1.6 How does the sum of the series $2 + 6 + 10$ compare with the value of the area of the rectangle?
2. Repeat the same procedure with the series $3 + 7 + 11 + 15$
 and $2 + 4 + 6 + 8 + 10$
 and then complete the summary sheet below:

Summary Sheet					
	1	2	3	4	5
Series	Width of Created Rectangle	Length of Created Rectangle	Area of Rectangle as Width x length	Sum of Series in terms of Area of Rectangle	Sum of terms of Series
$2+6+10$					
$3+7+11+15$					
$2+4+6+8+10$					
Describe in words a method for finding the sum of the first n terms of an arithmetic series					

WORKSHEET 6.2 ARITHMETIC SERIES

1. Several hundred years ago, a famous mathematician Gauss, as a young boy of approximately 9 years of age, was given the task of adding up the first hundred numbers.

He found a way of doing it so quickly and accurately that both his teacher and classmates were amazed.

He did not utilise the method outlined on worksheet 6.1.

What method do you think he used?

WORKSHEET 6.3 ARITHMETIC SERIES

1. Consider the sum of the first hundred numbers below:

$$S_{100} = 1 + 2 + 3 + \dots + 98 + 99 + 100$$

In the space provided below, rewrite $S_{100} = 1 + 2 + 3 + \dots + 98 + 99 + 100$ again but with the terms in reverse order

$$\text{i.e. } S_{100} = 100 + 99 + 98 + \dots + 3 + 2 + 1.$$

Then carefully consider the patterns you see and consider how you could find S_{100} from this representation.

2. Now consider the series $2 + 6 + 10 + \dots$ which we had before. Use the method used in question 1 to find the sum of the first 50 terms of the series.
3. Given the series $3 + 7 + 11 + \dots + 399$.
Find the sum of the series, using the method used in question 1.
4. Find the sum S_n for the series $2 + 6 + 10 + \dots$ of question 2.
5. Find the sum S_n for the series $3 + 7 + 11 + \dots + 399$ of question 3.
- 6.1 Find the sum of S_{10} for the series $a + (a + d) + (a + 2d) + \dots + 0$.
- 6.2 Find the sum of S_n for the series $a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$
Let $l = T_n = a + (n - 1)d$
- 6.3 Express your answer in 6.1 in terms of both $a; d$ and n
and also $a; l$ and n .

WORKSHEET 6.4 ARITHMETIC SERIES

1. Note the proof of our formulae

$$a + (a + d) + (a + 2d) + \dots + a + (n - 1)d = \frac{n}{2}[2a + (n - 1)d]$$

$$\text{i.e. } S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$\text{or } S_n = \frac{n}{2}(a + l)$$

Proof: let $a + (n - 1)d = l = T_n$

$$S_n = a + (a + d) + (a + 2d) + \dots + (l - d) + l \quad (1)$$

$$\& S_n = l + (l - d) + (l - 2d) + \dots + (a + d) + a \quad (2)$$

$$2S_n = [(a + l) + (a + l) + (a + l) + \dots + (a + l)] \times n \quad (1) + (2)$$

$$2S_n = n(a + l)$$

$$S_n = \frac{n}{2}(a + l)$$

$$S_n = \frac{n}{2}[a + a + (n - 1)d] \text{ from } T_n$$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Use these formulae to answer the following questions:

1. Calculate $\sum_{r=1}^{120} (4r - 3)$

2. Find the sum of the first 72 terms of a sequence with first term -6 and last term 222.

3. Solve for n if $\sum_{r=1}^n (6r - 1) = 320$.

WORKSHEET 6.5 ARITHMETIC SERIES

Write down the formula S_n you derived for the sum of the first n terms of the arithmetic series $a + (a + d) + (a + 2d) + \dots + a + (n - 1)d$ in the blocks below:

$S_n =$	or	$\sum_{r=1}^n [a + (r - 1)d] =$
---------	----	---------------------------------

Note, too, that if we let $l = a + (n - 1)d$, then

$$S_n =$$

$$=$$

i.e.

$S_n =$	or	$\sum_{r=1}^n \quad \quad \quad =$
---------	----	------------------------------------

Use the above formulae to help you answer the following questions:

1. Find the sum of the arithmetic series
 $1 + 7 + 13 + \dots$ to 100 terms.

2. Evaluate $3 - 7 - 17 - \dots$ to 70 terms.

3. Calculate $\sum_{r=1}^{36} (3r - 4)$.

4. What is the sum of the first 30 terms of a sequence with first term 0, 5 and last term 16, 5?

5. How many terms are there in the following series and what is the sum of the series?
 $2 + 5 + 8 + \dots + 74$.

6. Find the value of n if
 $\sum_{r=1}^n (6r + 28) = 1530$.

7. The first three terms of an arithmetic series are $-119; -110; -101$. If the sum of the series is -644 , how many terms are there in the series?

8. The sum of the first six terms of a finite arithmetic series is 144. If the sixth and last term is 39, determine the first term a and the common difference d of the series.

9. The sum of the first ten terms of an arithmetic series is $-4, 2$. The third and sixth terms add up to 1, 6. What would the value of the 12th term be?

10. The sum of the first 36 terms of an arithmetic series is 1530. The tenth term exceeds the sixth term by 16. Find the sum to 63 terms.
11. John increases the distance he runs by 16km per week. If he runs 90km in the first week, what is the total distance he will have run by the end of the 12th week?
12. What is the greatest value of n for which

$$\sum_{r=1}^n (4r - 3) < 276?$$
13. A businessman's gross income in his first year is R120 000-00 and his expenses are R40 000-00. Thereafter his income increases by R10 000-00 per year while his expenses increase by R6 000-00 per year.
- 13.1 How many years will it take for his net income to add up to R540 000-00?
- 13.2 Find his net income in his tenth year.
 (net income = gross income - expenses).
14. Determine the sum of the numbers between 1 and 200 which are divisible by 6.
15. The first three terms of an arithmetic series are $m - 2$, $2m - 6$ and $4m - 8$.
 Find:
- 15.1 m
- 15.2 T_{20}
- 15.3 S_{12}
16. Determine the sum of the numbers between 1 and 200 which are divisible by 6.

WORKSHEET 7.1 GEOMETRIC SERIES

1. Consider the sequence

1; 2; 4; -----

1; 2; 2^2 ; -----

- 1.1 What type of sequence is it?

Why?

The geometric series with $a = 1$ and $r = 2$

is $1 + 2 + 4 + 8 + \dots$

or $1 + 2 + 2^2 + 2^3 + \dots$

is called the binary series and its terms are called binary numbers.

- 1.2 See the diagrams below and complete the empty spaces:

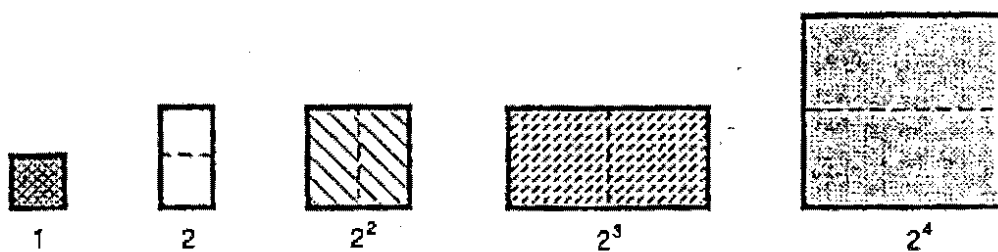


Fig. 1

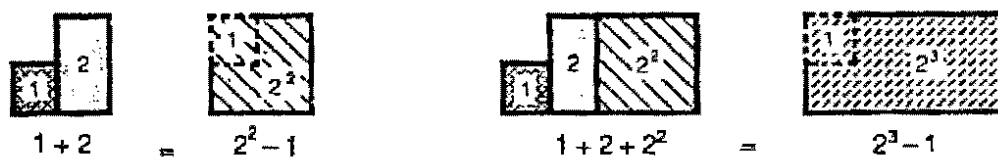


Fig. 2

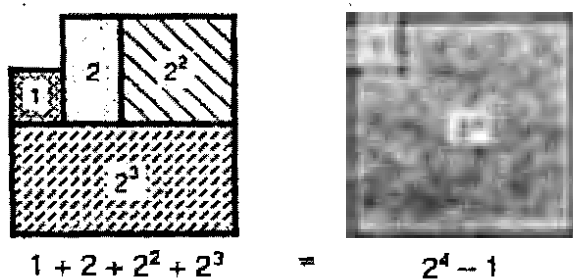


Fig. 3

(Bennett 1989: 131)

- 1.3 Complete the formula for the series below:

$$1 + 2 + 2^2 + \dots + 2^{n-1} =$$

2. Consider the sequence $1; 3; 9; \text{---}$
or $1; 3; 3^2; \text{---}$

2.1 What type of sequence is it?

Why?

2.2 See the diagrams below and complete the empty spaces.

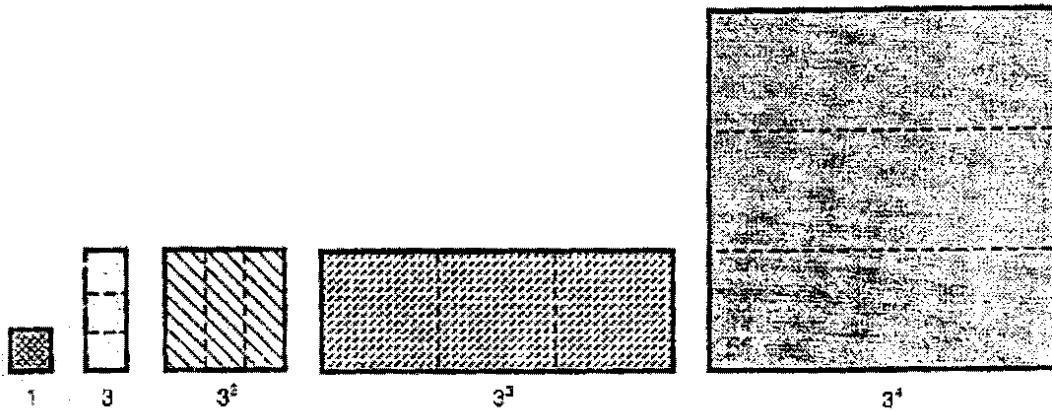


Fig. 4

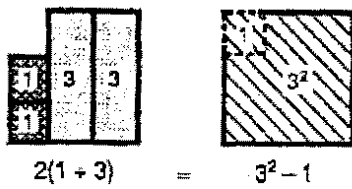


Fig. 5

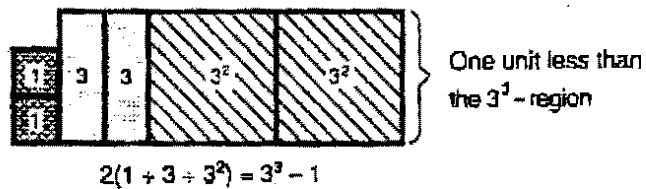


Fig. 6

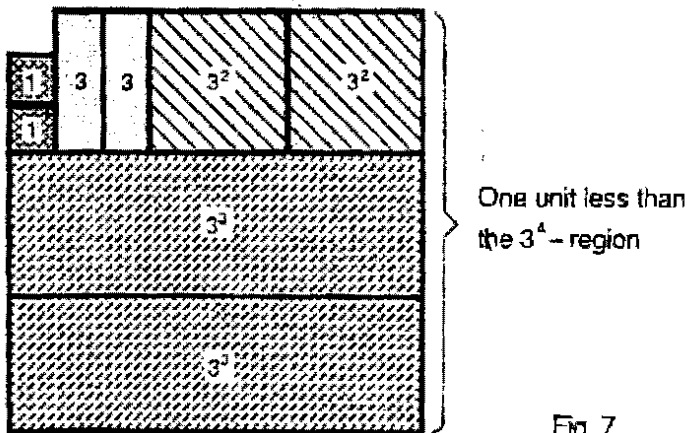


Fig. 7

(Bennet 1989: 131)

- 2.3 Complete the formula for the series below:
 $1 + 3 + 3^2 + \text{---} + 3^{n-1} =$

3. Consider the sequence 1; 4; 16; _____
or 1; 4; 4²; _____

3.1 What type of sequence is it?

Why?

3.2 See the diagrams below and complete the empty spaces.

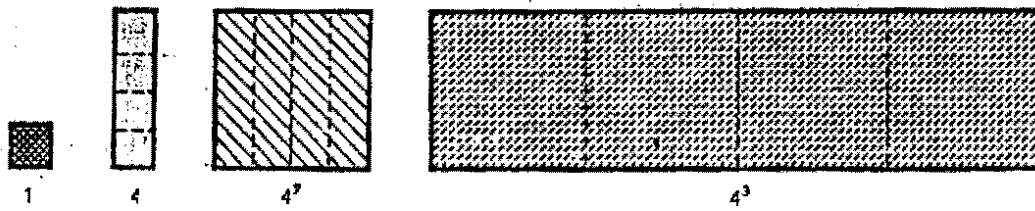


Fig. 8

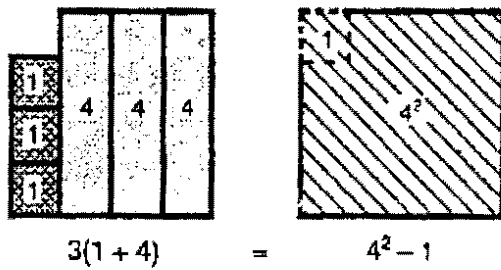


Fig. 9

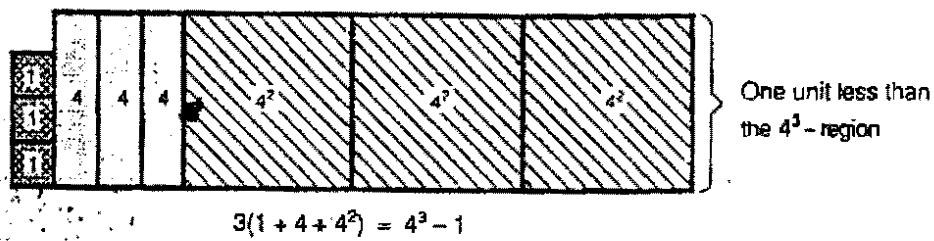


Fig. 10

(Bennett 1989: 132)

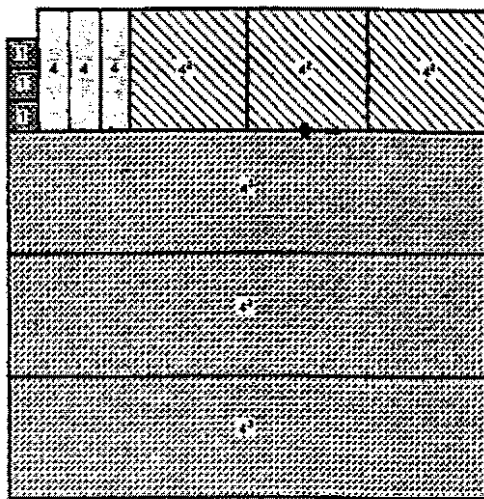


Fig. 11

(Bennett 1989: 133)

$$3(1 + 4 + 4^2 + 4^3) =$$

3.3 Complete the formula for the series below:

$$1 + 4 + 4^2 + \text{-----} + 4^{n-1} =$$

4.

4.1 The system of placing the regions next to each other as in the previous examples suggests a model for the general case. Study the diagram below..

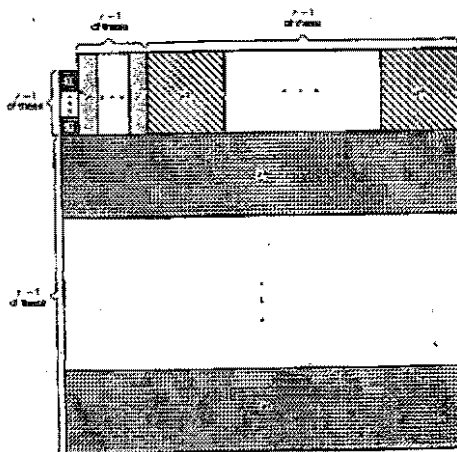


Fig. 12

(Bennett 1989: 133)

4.2 What series sum could be found from it?

4.3 Write down a formula you can deduce from it?

First, $(r-1)$ square units are placed in a rectangle; then $(r-1)$ r -regions are adjoined to form a square with one unit square missing. Next $(r-1)$ r^2 -regions are adjoined to form a rectangle with one unit square missing; then $(r-1)$ r^3 -regions are adjoined to form a square with one unit square missing; and so on; as shown in the previous figure. The area of each r^n -region is one square unit more than the total area of $(r-1)$ copies of each of the preceding regions. That is

$$(r-1)(1 + r + r^2 + \text{-----} + r^{n-1}) = r^n - 1,$$

and so

$$1 + r + r^2 + \text{-----} + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

WORKSHEET 7.2 GEOMETRIC SERIES

1. Consider the sequence 1; 3; 9; ————
 - 1.1 What sort of sequence is it?
 - 1.2 Write down the corresponding series:
 - 1.3 Try to think of a way of finding the sum of the first seven terms of the series without adding up term by term.

WORKSHEET 7.3 GEOMETRIC SERIES

1. In the geometric series $1 + 3 + 9 + \dots$ of the previous worksheet, what are a and r ?
2. Write down S_7 and rS_7 where rS_7 is the series obtained by multiplying each term of the series S_7 by r .
3. What pattern do you notice if rS_7 is written below S_7 with like terms underneath each other?
4. How could you use what you have observed in 3 to help you find S_7 without adding up term by term?
5. Check your answer obtained in 4 on your calculator.

WORKSHEET 7.4 GEOMETRIC SERIES

1. What are a and r in the geometric series $2 + 4 + 8 + \dots$?

Use the method of the previous worksheet to help you find the sum to ten terms of the series.

2. What are a and r in the geometric series $1 - 4 + 16 - \dots$?
Use the method of the previous worksheet to help you to find the sum to seven terms of the series.
3. Find a formula representing S_n for the series $2 + 4 + 8 + \dots$ of question 1.
4. Find a formula representing S_n for the series $1 - 4 + 16 + \dots$ of question 2.
5. Derive a formula representing S_n for the sum to n terms of the general geometric series.
 $a + ar + ar^2 + \dots + ar^{n-1}$
6. Could you think of another way of writing down the formula you have derived in 5?

WORKSHEET 7.5 GEOMETRIC SERIES

1. Note the proof of our formulae
 $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{a(1 - r^n)}{1 - r}$ below:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}$$

$$\begin{aligned} \text{Proof: } S_n &= a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} & (1) \\ \text{xr } S_n &= ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} + ar^n & (2) \\ S_n - rS_n &= a - ar^n & (1) - (2) \\ S_n(1 - r) &= a(1 - r^n) \\ S_n &= \frac{a(1 - r^n)}{1 - r} \text{ or } \frac{a(r^n - 1)}{r - 1} & (2) - (1) \end{aligned}$$

Use these formulae to answer the following questions:

1. Calculate $\sum_{k=1}^{21} 2 \cdot 3^k$
2. Find the sum of the first 30 terms of the series:
 $2 - 1 + \frac{1}{2} - \dots$
3. Calculate the value of m if
 $\sum_{k=1}^m \frac{1}{5}(3)^{k-1} = 24\frac{1}{5}$

WORKSHEET 8.1 INFINITE SERIES

1. If $T_k = \frac{1}{k}$, find the value of T_k as k becomes very large i.e. as k tends to ∞ , written as $k \rightarrow \infty$
we may write $\lim_{k \rightarrow \infty} T_k =$ _____

This means the limit at k tends to infinity or the value which the terms of the sequence are approaching as more and more terms of the sequence are taken.

2. If $T_k = -\frac{2}{k^2}$, find the value which the terms of the sequence are approaching as k becomes very large i.e. as $k \rightarrow \infty$ (k tends to infinity).
We may write : $\lim_{k \rightarrow \infty} T_k =$ _____

3. Recall the sequence of numbers
1; 1; 2; 3; 5; 8; 13; 21; 34; -----
3.1 What is this sequence called?

- 3.2 Calculate the first thirty terms of the sequence.

- 3.3 Evaluate $\frac{T_2}{T_1}, \frac{T_3}{T_2}, \frac{T_4}{T_3}, \dots, \frac{T_{k+1}}{T_k}, \dots$
i.e. $T_2 \div T_1; T_3 \div T_2; T_4 \div T_3, \dots, T_{k+1} \div T_k, \dots$

- 3.4 Do you think $\frac{T_{k+1}}{T_k}$ tends to a limit as $k \rightarrow \infty$?

- 3.5 Evaluate $\frac{\sqrt{5}+1}{2}$.

- 3.6 What do you notice?

The number $\Phi = \frac{\sqrt{5}+1}{2}$ is called the Golden Ratio and has a value of approximately 1,618 when expressed in decimal form.

The Golden Ratio Φ was well known to the ancient Greeks. Here Φ stands for Phidias who was the greatest sculptor of antiquity. Φ is closely related to the Fibonacci numbers. The ancient Greeks were fascinated by the aesthetic proportion generated by Φ in geometry art and architecture. Tests have revealed that rectangles for which the ratio of the base to the height is Φ are most pleasing to the eye.

- 3.7 We see that as $k \rightarrow$ _____

$$\frac{T_{k+1}}{T_k} \rightarrow \text{_____}$$

- i.e. $\lim_{k \rightarrow \infty}$ _____

4. Consider the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \dots$$

4.1 What is the rule for it?

4.2 First find out what value the terms of the two sequences below approach by taking more and more terms.

$$4.2.1 \quad \frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \frac{1393}{985}, \dots$$

$$4.2.2 \quad \frac{3}{2}, \frac{17}{12}, \frac{99}{70}, \frac{577}{408}, \dots$$

4.3 Find the value of the irrational number $\sqrt{2}$.

4.4 What can you conclude?

4.5 Complete

$$\underline{\hspace{2cm}} < \sqrt{2} < \underline{\hspace{2cm}}$$

5. Consider the series

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

5.1 By increasing the number of terms indefinitely, i.e. by taking more and more terms, do you find the result tends to a limit?

5.2 Find the value of the irrational number π .

5.3 What do you notice?

5.4 What can you conclude?

$$\underline{\hspace{2cm}} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

6. $1!$ means 1
 $2!$ means 2×1
 $3!$ means $3 \times 2 \times 1$
 $4!$ means _____
 $5!$ means _____
 $6!$ means _____
 $n!$ means _____

Now consider the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots$$

6.1 Does this series appear to approach a fixed value?

6.2 Find the value of the irrational number e .

6.3 What can you conclude?

LXIV

7. By summing more and more terms of the following series
i.e, letting $n \rightarrow \infty$, calculate the following sums:

7.1 $7 + 7 + 7 + \dots$

7.2 $1 + 2 + 4 + \dots$

7.3 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2n}$

7.4 $2 + 4 + 6 + \dots + 2n + \dots$

7.5 $1 - \frac{1}{3} + \frac{1}{9} \dots + (-\frac{1}{3})^{n-1}$

7.6 $4 - 1 - 6 \dots$

Do you think it is ever possible to find S_∞ for a series?
If you think it is, for what types of series can it be done?

WORKSHEET 8.2 LIMITS

1. We shall first consider geometric series for which $a = 1$ and beginning with the second term.

$$\text{i.e. } a + ar + ar^2 + ar^3 + \dots$$

$$\text{becomes } r + r^2 + r^3 + \dots$$

If we let $r = \frac{1}{k}$ where k is a whole number greater than 1, then this becomes

$$\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \dots$$

If $k = 2$, then we get

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

Now consider the sum of the first few terms of the series as follows:

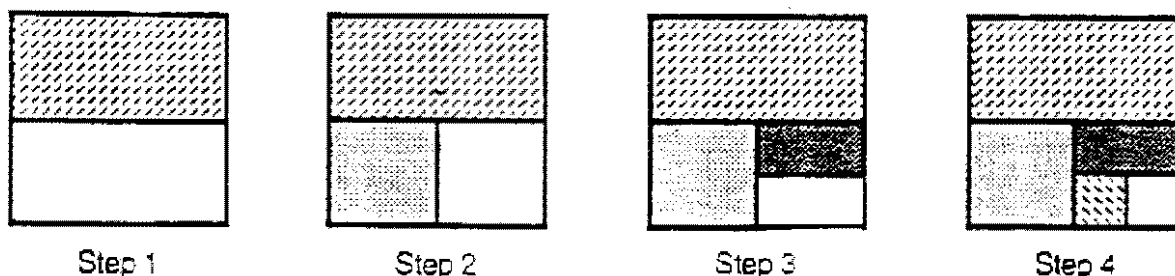


Figure 13.

(Bennett 1989: 134)

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

- 1.1 What do you think will be the value of $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

- 1.2 Compare your answer with Worksheet 8.1 Q7.3.

1.3

Four copies of a unit square are shown in figure 13. In step 1 the square is divided into halves, and one half is shaded and one half remains for further subdividing. In step 2 the remaining part is divided into halves, and one part is shaded and the other part remains for further subdividing, and so on.

If this subdivision process is continued, it produces a sequence of shaded parts, each of which is one-half the size of the preceding part, and the total shaded region represents

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

Since the size of the remaining part at each step approaches zero, the total shaded region approaches one whole square. That is,

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 1$$

2. Now consider the series

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$$

i.e. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$

illustrated below.

What do you think the value of $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$ will be? _____

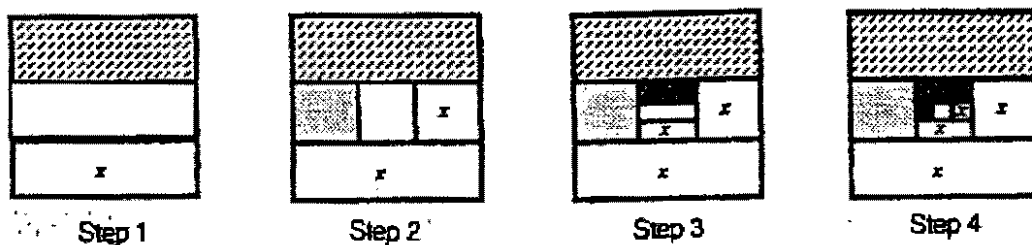


Fig. 14

(Bennett 1989: 134)

Figure 14 contains four copies of a unit square. In step 1 the square is divided into thirds, and one part is shaded, one part is marked with an x , and one part is left unmarked for further subdividing. In step 2 the unmarked part is divided into thirds, and one part is shaded, one part is marked by x , and one part is left unmarked for further subdividing, and so on.

If this process is continued, it produces a sequence of shaded parts each of which is one-third the size of the preceding part, and the total shaded region represents

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots$$

Since the area of the unmarked part of each step approaches zero, and for each shaded part is a corresponding x -marked part of equal size (the ratio of shaded parts to x -marked parts is 1 to 1), the shaded region approaches one half of the squares. therefore,

$$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots = \frac{1}{2}$$

3. If $k = 4$, we have the series

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots$$

i.e. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

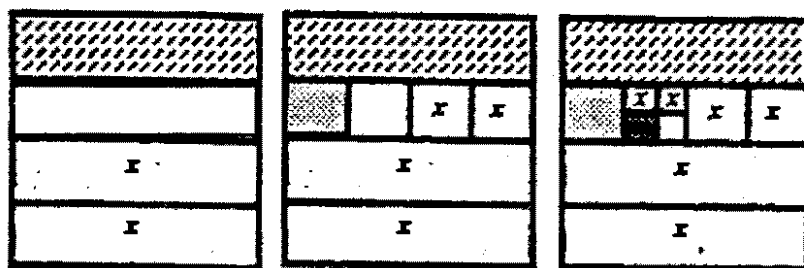


Fig. 15

(Bennett 1989: 135)

Figure 15 shows the first three steps in the process of subdividing a unit square into parts of decreasing size, each of which is one-fourth the size of the preceding part. In step 1 the square is divided into fourths, and one part is shaded, two parts are marked with x, and one part is left unmarked for further subdividing. In step 2 the remaining part is shaded, two parts are marked with x, and one part is left unmarked for further subdividing, and so forth.

If this process is continued, it produces a sequence of shaded parts whose total shaded region represents

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots$$

Since the size of the unmarked part each step approaches zero, and for each shaded part are two corresponding x-marked parts of equal size (the ratio of shaded parts to x-marked parts is 1 to 2), the shaded region approaches one-third of the square. So,

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{1}{3}$$

What do you think the value of

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots \text{ will be?}$$

4. Guess the values of the sums of the following series:

4.1 $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots$

4.2 $\frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \dots$

4.3 $\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \dots$

4.4 $\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \dots$

WORKSHEET 8.3 INFINITE ARITHMETIC AND GEOMETRIC SERIES

1. Recall that the sum of S_n of an arithmetic series is given by

or

and a geometric series is given by

or

- 1.1 Consider the arithmetic series

$$2 + 5 + 8 + \dots$$

$$S_1 =$$

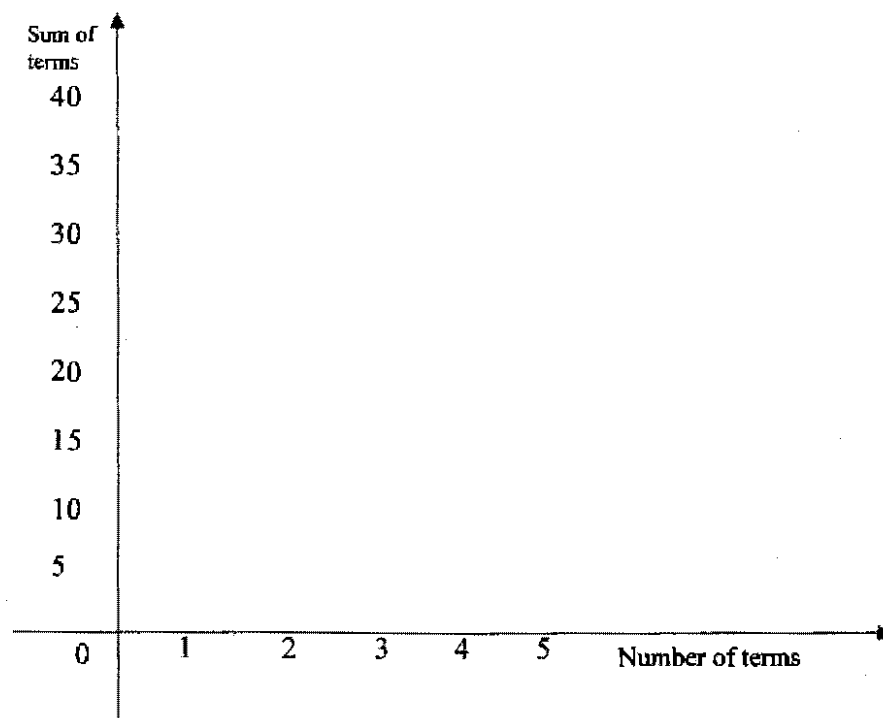
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:



1.2 Now consider the arithmetic series:

$$1 - \frac{1}{2} - 2 \dots \dots \dots$$

$$S_1 =$$

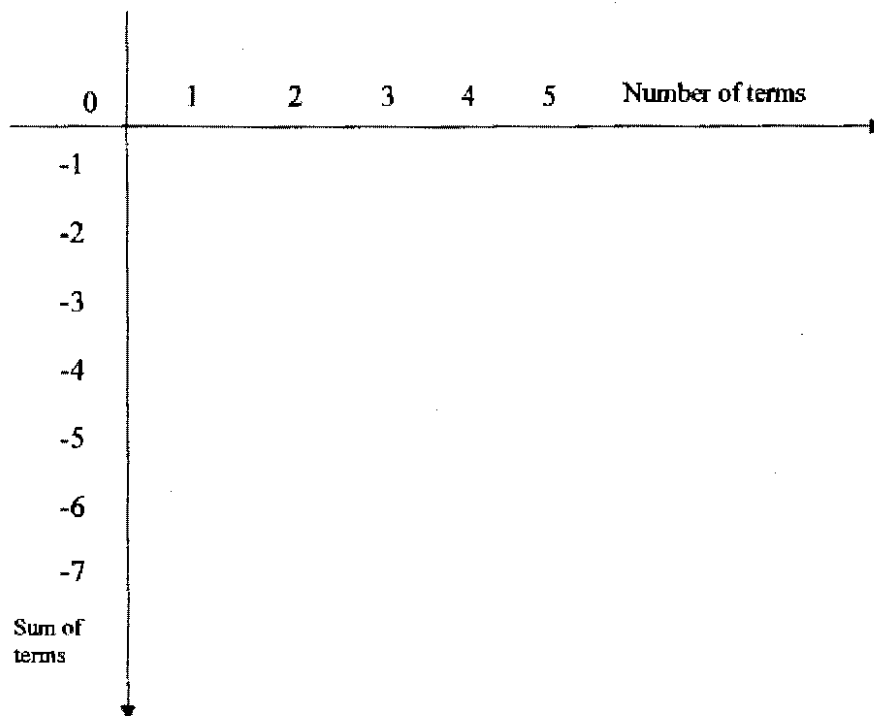
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:



We can conclude that:

as $n \rightarrow \infty, S_n \rightarrow$ _____

i.e. $\sum_{n=1}^{\infty}$ _____ \rightarrow _____

2. Now consider the following cases of geometric series:

2.1 $|r| < 1$ i.e. $-1 < r < 1$

e.g. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$$S_1 =$$

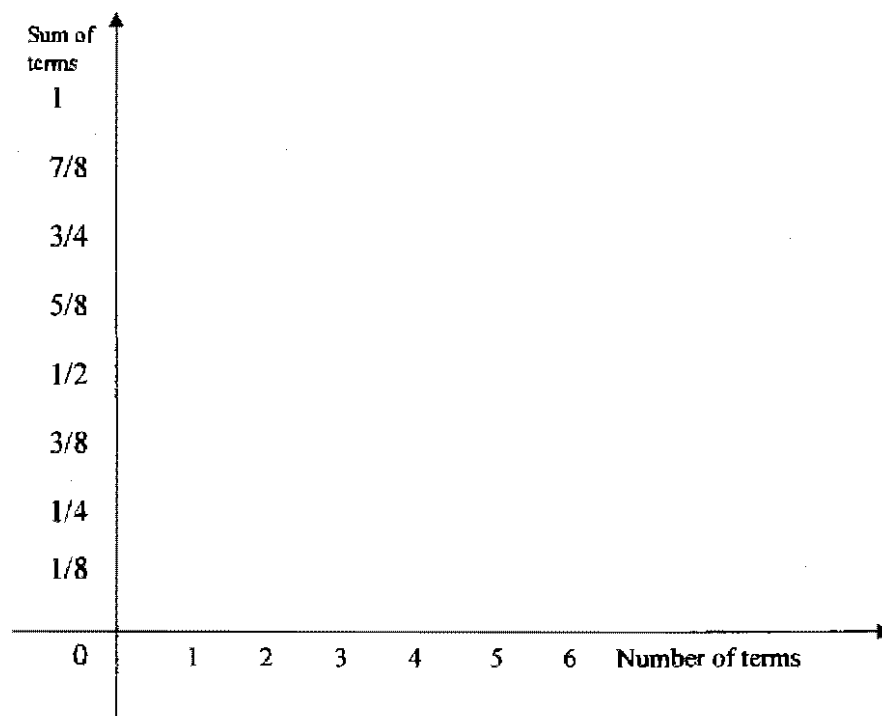
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:



(Laridon 1996: 89)

We can conclude that:

as $n \rightarrow \infty$, $S_n \rightarrow$ _____

i.e. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n =$ _____

2.2 $r > 1$
 e.g. $1 + 3 + 9 + \dots$

$$S_1 =$$

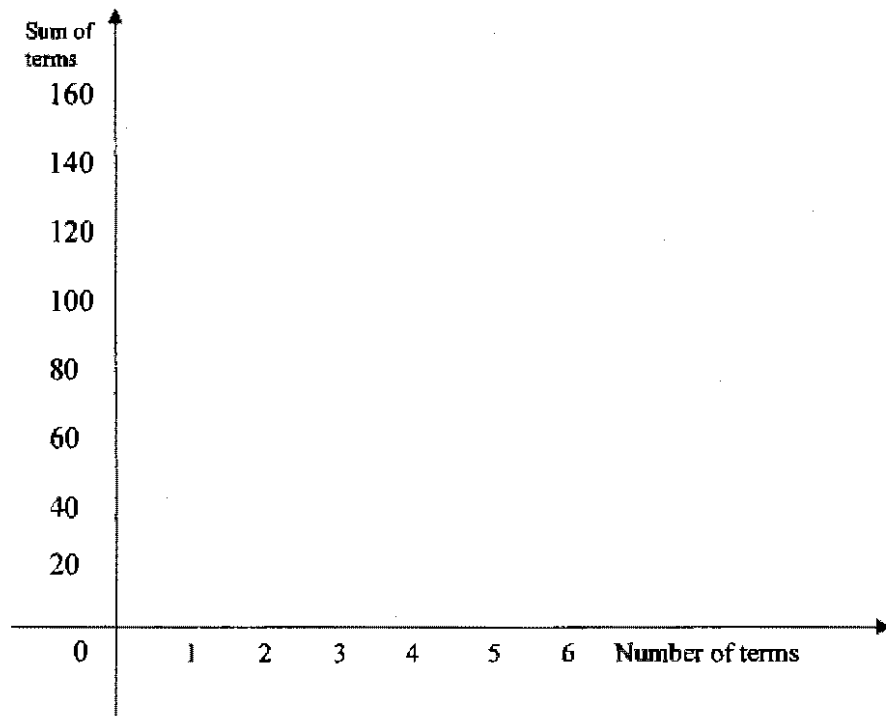
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:



(Laridon 1996: 90)

What do you think happens as $n \rightarrow \infty$?

i.e. as $n \rightarrow \infty$, $S_n \rightarrow$ _____

2.3 $r < 1$
 e.g. $1 - 3 + 9 - 27 + 81 + \dots$

$$S_1 =$$

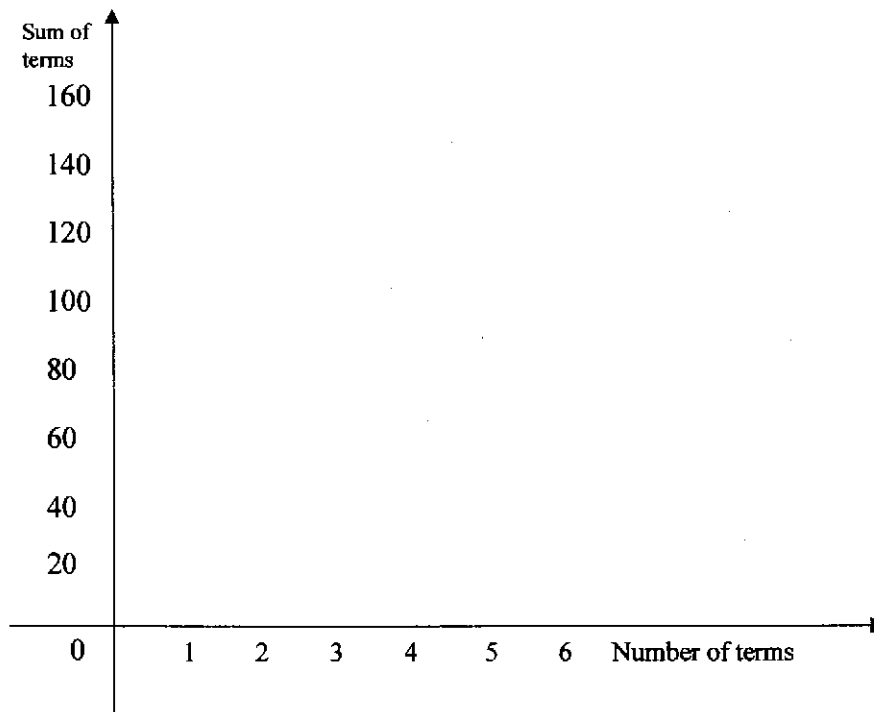
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:



What do think happens as S_n as $n \rightarrow \infty$?

i.e. as $n \rightarrow \infty$, $S_n \rightarrow$ _____

2.4 $r = +1$

2.4.1 For the series:

 $3 + 3 + 3 + \dots$

$$S_1 =$$

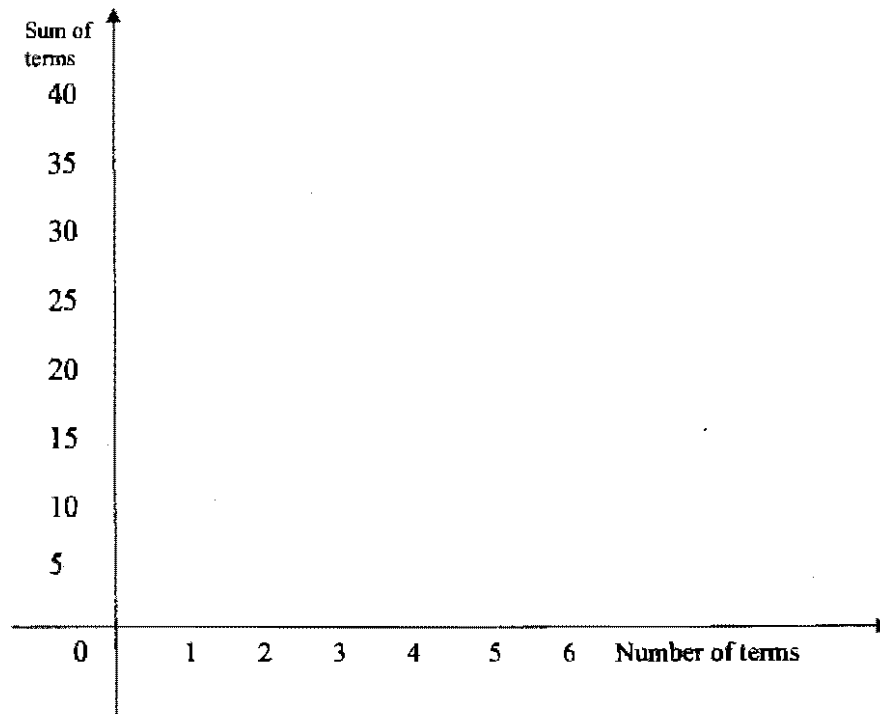
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:

What do you think happens to S_n as $n \rightarrow \infty$?i.e. $n \rightarrow \infty$, $S_n \rightarrow$ _____

2.4.2 For the series:

$$3 - 3 + 3 - 3 + \dots$$

$$S_1 =$$

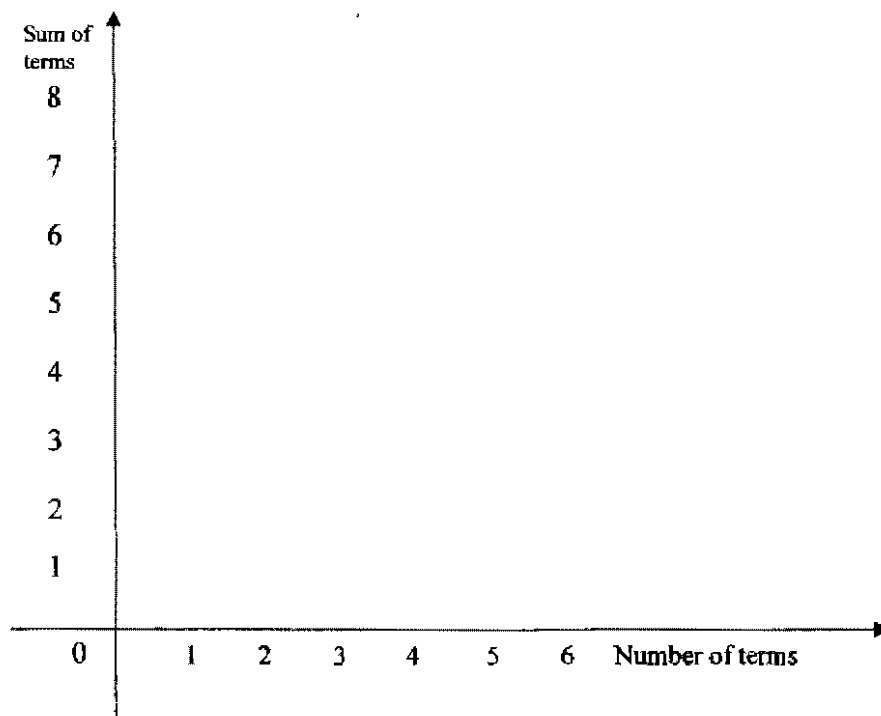
$$S_2 =$$

$$S_3 =$$

$$S_4 =$$

$$S_5 =$$

Plot the values on the graph below:

What do you think happens to S_n as $n \rightarrow \infty$?i.e. $n \rightarrow \infty$, $S_n \rightarrow$ _____

3.

3.1 Do you think it is possible to evaluate S_∞ for arithmetic series: always, sometimes or never? Explain.

3.2 Do you think it is possible to evaluate S_∞ for geometric series: always, sometimes or never? Explain.

WORKSHEET 8.4 GEOMETRIC SERIES

Consider the formula

$$\begin{aligned}
 S_n &= \frac{a(1-r^n)}{1-r} \\
 &= \frac{a-ar^n}{1-r} \\
 &= \frac{a}{1-r} - \frac{ar^n}{1-r} \\
 &= \frac{a}{1-r} - \frac{a \cdot r^n}{1-r}
 \end{aligned}$$

Suppose $|r| < 1$ i.e. $-1 < r < 1$.

e.g. $r = \pm \frac{1}{2}$, etc

What will happen to the value of r^n as $n \rightarrow \infty$?

Hence can you find a formula for S_∞ ?

$S_\infty =$

Does this formula hold only for $|r| < 1$?

Consider the other cases too.

1. Use your new formula to find the sum of the following series:

1.1 $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

1.2 $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots$

1.3 $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots$

1.4 $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \dots$

1.5 $\frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \dots$

1.6 $\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} + \dots$

Do your answers confirm the answers you obtained in Worksheet 8.2?

2. Calculate $\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n$
3. Find the geometric series which has a first term of 144 and a sum to infinity of 108.
4. For which values of x will the infinite series $1 + \frac{1}{2x-1} + \frac{1}{(2x-1)^2} + \dots$ converge?
5. If a tree grows a height of $2m$ in its first year and each year thereafter $\frac{3}{4}$ of it's previous height, how tall can it grow?

Worksheet 9.1 Sequences and Series

1. Expand $(x + y)^n$ for the values $n = 0; 1; 2; 3; 4$.
2. Do you see any pattern? Explain.
3. Try to find a relationship between the coefficients in each answer and those in the previous one.
4. Study the following triangle called Pascal's triangle.
Search for a pattern and try to fill in the next four rows.

1

1 1

1 2 1

5. Can you see a relationship between your answers in questions 1 and 4?

6. Use Pascal's triangle to expand

6.1 $(x + y)^4$

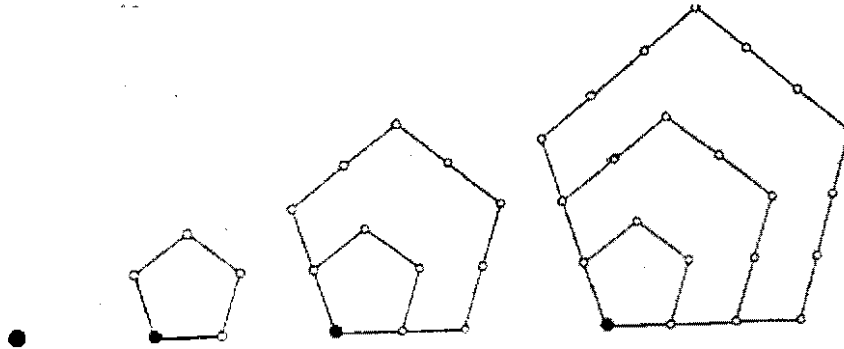
6.2 $(a - b)^5$

6.3 $(2m - n^2)^6$

Worksheet 9.2 Sequences and Series

1. The Pythagoreans were particularly interested in the shape of five and the pentagonal numbers.

There is a pattern to be found in the number of vertices of the following sequence of pentagonal numbers.



(Reid 1992: 77)

- 1.1 Complete:

$$S_1 = 1$$

$$S_2 = 1 + 4 =$$

$$S_3 =$$

$$S_4 =$$

- 1.2 What sort of series is formed each S_n ?

- 1.3 How many vertices are there in the 12th pentagon of the sequence?

- 1.4 Can you predict how many vertices there would be in the n th pentagon?

2. there are many patterns which arise from sequences of numbers.

- 2.1 For example, consider the sum of consecutive odd numbers, as follows:

$$1 =$$

$$1 + 3 =$$

$$1 + 3 + 5 =$$

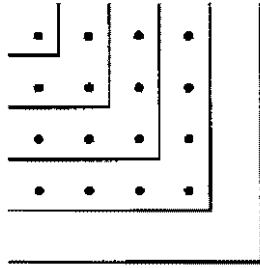
$$1 + 3 + 5 + 7 =$$

$$1 + \dots\dots\dots$$

$$1 + \dots\dots\dots$$

- 2.2 What do you think the rule is here?

- 2.3 Complete the following diagram and see whether it illustrates your results in 2.1 and 2.2.



2.4 Do you think this result has been proved or needs to be proved?

3.

3.1 Now complete the sum of the consecutive even numberfs as follows:

$$2 =$$

$$2 + 4 =$$

$$2 + 4 + 6 =$$

$$2 + \dots\dots\dots$$

$$2 + \dots\dots\dots$$

$$2 + \dots\dots\dots$$

3.2 Could you predict the sum of the first twenty even numbers?
If so, find it.

3.3 Could you predict the sum of the first n numbers?
If so, find it.

3.4 Do you think this result has been proved / could be proved?

4. Consider the sequence with $T_n = n^2 - n + 41$.
We wish to investigate whether every term will be a prime number.

4.1 Find the first ten terms.
i.e. Complete the following table:

n	$n^2 - n + 41$
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

(Fleming & Varberg 1989: 427)

4.2 Are all your answers prime?
(A prime number is a number with exactly two factors, itself and one.)

- 4.3 Is T_n a prime number for all n ?
- 4.4 Find the value of T_{41} .
- 4.5 Do you need to prove a result or can we conclude it must be true if it is true for the first ten or so values? Explain.

Worksheet 9.3 Mathematical Induction

1.

Let P_n represent the statement $n^2 - n + 41$
that $n^2 - n + 41$ is a prime number.

i.e. $P_n : n^2 - n + 41$ is a prime number.

1.1 Is P_1 true?1.2 Is P_2 true?1.3 Is P_3 true?1.4 Is P_n true for all n ?

A statement may be regarded as a sentence which is either true or false. The principle of mathematical induction deals with a sequence of statements. In a sequence of statements, we have a statement corresponding to each positive number as we saw in the example above.

Consider the following 3 examples too:

$$Q_n : (a + b)^n = a^n + b^n$$

$$R_n : 1 + 2 + 3 + \dots + n = \frac{n^2 + n - 6}{2}$$

$$S_n : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(Fleming & Varberg 1989: 427)

2.

2.1 Is Q_1 true?2.2 Is Q_2 true?2.3 Is Q_n true?

3.

3.1 Is R_1 true?3.2 Is R_2 true?3.3 Is R_n true?

4.

4.1 Is S_1 true?4.2 Is S_2 true?4.3 Is S_n true?5.1 Do we need to prove that S_n is true for all n ?5.2 How could we prove S_n is true for all n ?

Worksheet 9.4 Mathematical Induction

1. Mathematical Induction is the tool uniquely designed for proving that statements are true for all n .

In order to prove that statement S_n is true for all n by Mathematical Induction we must prove that it is true for $n = 1$.

Assuming that it is true for some $n = k$, we must show that it is true for $n = k + 1$. It then follows by Mathematical Induction that it is true for all natural values of n .

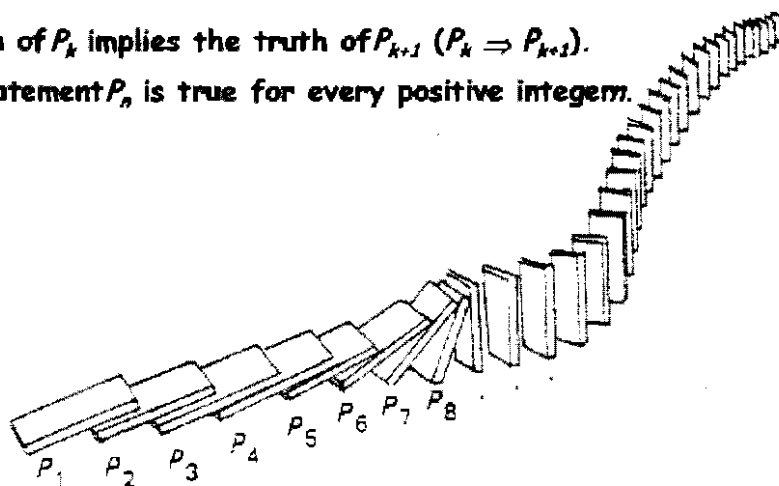
To elaborate on this method, we could consider the analogy of a row of dominoes (assuming there are infinitely many) standing up and spaced equally far apart as follows:

The principle of Mathematical Induction.

Let P_1, P_2, P_3, \dots Be a sequence of statements with the following two properties.

1. P_1 is true.
2. The truth of P_k implies the truth of P_{k+1} ($P_k \Rightarrow P_{k+1}$).

Then the statement P_n is true for every positive integer n .




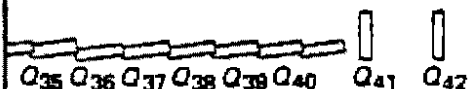
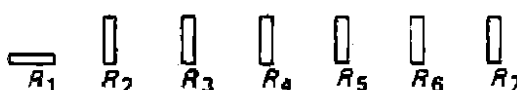

(Fleming & Varberg 1989: 427)

Using the idea of the dominoes here, we note that for all dominoes to fall it is sufficient that

1. the first domino is pushed over.
2. if any domino falls (say the k th one), it pushes over the next one (the $(k + 1)$ th one).

Consider the following illustrations of dominoes regarding the four examples we had in the previous worksheet.

Why They Fail and Why They Don't

$P_n : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ <p>P_1 is true</p> <p>$P_k = P_{k+1}$</p>	 <p>First domino is pushed over. Each falling domino pushes over the next one.</p>
$Q_n : n^2 - n + 41 \text{ is prime.}$ <p>Q_1, Q_2, \dots, Q_{40} are true.</p> <p>$Q_k \neq Q_{k+1}$</p>	 <p>First 40 dominoes are pushed over. 41st domino remains standing.</p>
$R_n : (a+b)^n = a^n + b^n$ <p>R_1 is true.</p> <p>$R_k \neq R_{k+1}$</p>	 <p>First domino is pushed over but dominoes are spaced too far apart to push each other over.</p>
$S_n : 1 + 2 + 3 + \dots + n = \frac{n^2 + n - 6}{2}$ <p>S_1 is false.</p> <p>$S_k = S_{k+1}$</p>	 <p>Spacing is just right but no one can push over the first domino.</p>

(Fleming & Varberg 1989: 429)

We shall need to show that S_n is true for all n i.e. that S_1 is true and that $S_k \Rightarrow S_{k+1}$

$$S_n : \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

$$S_1 : \begin{aligned} LHS &= \frac{1}{1 \cdot 2} = \frac{1}{2} \\ RHS &= \frac{1}{1+1} = \frac{1}{2} \end{aligned}$$

Thus S_1 is true.

Assume S_k is true for some k .

$$\text{i.e. assume that } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We must now show that S_{k+1} is true.

$$\text{i.e. that } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\text{i.e. } \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Now,

$$\begin{aligned}
 LHS &= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
 &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
 &= \frac{k(k+2)+1}{(k+1)(k+2)} \\
 &= \frac{k^2+2k+1}{(k+1)(k+2)} \\
 &= \frac{(k+1)^2}{(k+1)(k+2)} \\
 &= \frac{k+1}{k+2} \\
 &= RHS
 \end{aligned}$$

Thus the truth of S_k implies the truth of S_{k+1} .

Hence S_n is true for all positive integers n .

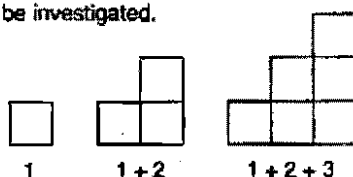
Worksheet 9.5 Visual Approach to Induction

INVESTIGATING CONSECUTIVE NATURAL NUMBERS

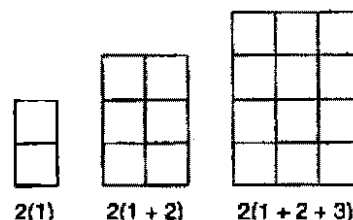
SHEET 1A

On this sheet the sums of consecutive natural numbers will be investigated.

1. On graph paper, copy the figures at the right, draw the next three diagrams in the sequence, and write their sums below each diagram.



2. For each diagram given in problem 1, its duplicate has been drawn above it to make a rectangle, as shown at the right. Do the same for the three diagrams drawn by you in problem 1, altering appropriately the sum underneath.



3. The dimensions of the rectangles drawn in problem 2 are 1×2 , 2×3 , 3×4 , and so on. Place the dimensions of each rectangle on the graph paper above the figure. Do the arithmetic to check that simplifying the expression above and below each rectangle leads to the same number.
4. Complete the following chart without drawing figures for $n = 7, 10, 12$, or k . Look for a pattern. Make a conjecture. Check the two values to see if the conjecture is correct. Fill in the final entry, for any number k , once the pattern is understood.

Value of n	Value of $2(1 + 2 + \dots + n)$	Dimensions of Rectangle Formed by Two Blocks	Are Columns 2 and 3 Equal?
3			
4			
5			
6			
7			
\vdots			
10			
\vdots			
12			
\vdots			
k			Cannot check, but observed this pattern.

5. Without using the formula, explain why $k(k+1)/2$ is an integer whenever k is an integer.

(Van Dyke 1995: 305)

Consider the formula $2(1 + 2 + 3 + \dots + n) = n(n + 1)$, where n is any natural number. Complete the following.

The formula for $n = 9$ is _____ = _____.

The formula for $n = 6$ is _____ = _____.

The formula for $n = 2$ is _____ = _____.

Check to see that the formula is true in all these cases.

We shall prove that the formula is true for all natural numbers n using the principle of mathematical induction.

Step 1

Write the formula for $n = 1$. _____ = _____. Show that it gives a true statement.

Step 2

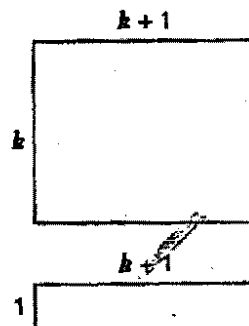
Write the formula for $n = k$. _____ = _____.

Write the formula for $n = k + 1$. _____ = _____.

Show that if _____ = _____ (the formula for $n = k$),

then _____ = _____ (the formula for $n = k + 1$).

- a) The product on the right side of the case $n = k$ can be pictured as the area of the rectangle at the right. This rectangle has dimensions k by $k + 1$.



- b) The rectangle with dimensions $k + 1$ by 1 has an area of $k + 1$ square units.

- c) Create a large rectangle using the rectangle in (a) along with two rectangles of the type shown in (b). Draw the resulting figure. What are the dimensions of the figure? Is this figure a representation for the right side of the formula for $n = k + 1$?

- d) Complete the induction proof. First write the formula for $n = k$. _____ = _____.

Next add $2(k + 1)$ to both sides. _____ = _____.

Manipulate and factor to obtain the formula for the case $n = k + 1$ as in step 2.

_____ = _____.

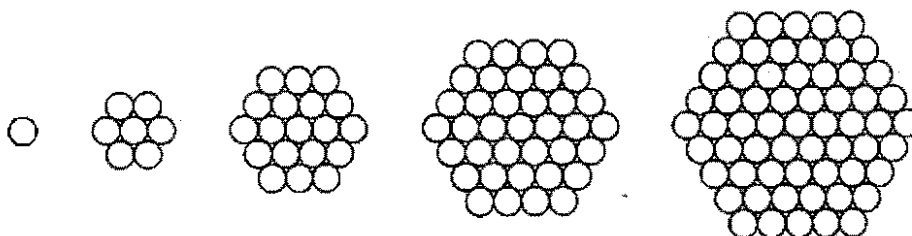
- e) Explain why step 1 together with step 2 renders the conclusion that the formula is true for all natural numbers.

From the *Mathematics Teacher*, April 1996

(Van Dyke 1995: 306)

Worksheet 9.6 Mathematical Induction

1. Prove by Mathematical Induction that
 $2 + 4 + 6 + \dots + 2n = n(n + 1).$
2. Prove by Mathematical Induction that
 $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. Consider the sequence represented below:



(Jacobs 1994: 94)

- 3.1 Find the general term T_k for this sequence.
- 3.2 Find the formula for S_n if this is true.
- 3.3 Prove your result in 3.2 by means of Mathematical Induction.

APPENDIX B

Questionnaires used in investigation component

Questions on Sequences and Series

Questionnaire 1

Name: _____

Please answer the following questions in the spaces provided:

1. How would you describe
 - 1.1 a pattern

 - 1.2 a sequence

2. Did the use of visual representations (blocks, pictures, etc) help you to understand the meaning of
 - 2.1 A sequence? Explain.

 - 2.2 The n th term of a sequence? Explain.

3. Did the study of patterns help you to understand the meaning of numbers that follow:
 - 3.1 A sequence? Explain.

 - 3.2 The n th term of a sequence? Explain.

4. What have you found to be the most
 - 4.1 difficult thing covered so far? Why?

 - 4.2 easy thing covered so far? Why?

5. Do you have an idea of how the concept of a sequence first came about?
How?
6. Do you see any use for the n th term of a sequence? Why?
7. Give a summary of what you have learned so far in these lessons.

Questions on Sequences and Series

Questionnaire 2

Name: _____

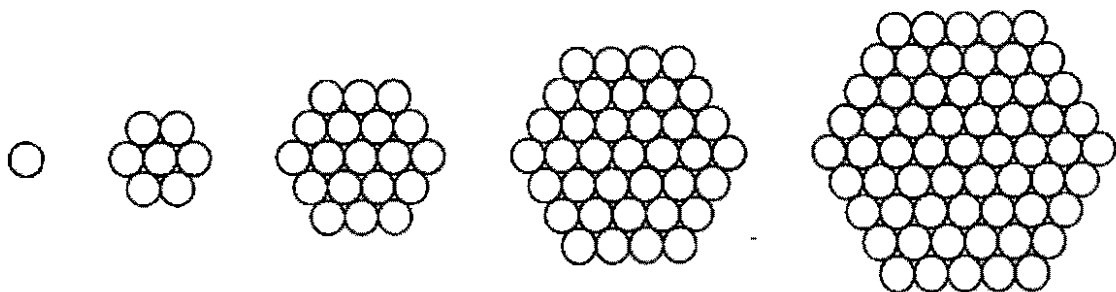
Please answer the following questions in the spaces provided:

1. What different properties do all patterns have in common?

2. What different properties do all sequences have in common?

3. Are patterns always immediately obvious? Explain.

- 4.



(Jacobs 1994: 94)

- 4.1 What aspect(s) of the visual representation above help(s) us to identify and continue the sequence?

- 4.2 How would we arrive at a formula for the n th general term of this sequence?

- 5.

- 5.1 What rule did we establish for the k th term of a Fibonacci sequence?

- 5.2 Did it help us predict the 100th term easily? Explain.
6. Explain the meaning of the k th term, the n th term and the general term of a sequence.
- 7.
- 7.1 What is an arithmetic sequence?
- 7.2 What rule did we establish for the general term of an arithmetic sequence?
- 8.
- 8.1 How many variables are there in the formula for the general term of an arithmetic sequence?
- 8.2 Why are the different variables necessary in the general term of an arithmetic sequence and what do they each represent?
9. Do you think it is useful to find the general term of an arithmetic sequence? Explain.
- 10.
- 10.1 Explain the difference between $T_{20} = r$ and $T_r = 20$.
- 10.2 Find an example of a sequence for which
- 10.2.1 $T_{20} = r$
- 10.2.2 $T_r = 20$

10.2.2 $T_r = 20$

11.

11.1 Did visualisation help you to do the examples on worksheets 2.5 and 3.4? Explain.

11.2 Did exploring patterns help you to do the examples on worksheets 2.5 and 3.4? Explain.

12. Do you think that the topic of arithmetic sequences is relevant? Explain.

Questions on Sequences and Series

Questionnaire 3

Name: _____

Please answer the following questions in the spaces provided:

1. What is

1.1 visualisation?

1.2 a sequence?

1.3 an arithmetic sequence?

1.4 a geometric sequence?

1.5 a Fibonacci sequence?

2.

2.1 Are all sequences arithmetic, geometric or Fibonacci? Explain.

2.2 Are all arithmetic, geometric and Fibonacci sequences examples of sequences? Explain.

3.

3.1 What is the use of a general term of a sequence?

4.
 - 4.1 Write down the general term of an arithmetic sequence with the first term a and common difference q .
 - 4.2 Write down the general term of a geometric sequence the first term a and common ratio q .
 - 4.3 Compare your answers in 4.1 and 4.2 carefully, noting any similarities or differences.
5.
 - 5.1 Give an example of a sequence which is both arithmetic and geometric.
 - 5.2 Use the formula for the general term of an arithmetic sequence to find the general term for your sequence in 5.1.
 - 5.3 Use the formula for the general term of a geometric sequence to find the general term for your sequence in 5.1.
 - 5.4 Compare your answers in 5.2 and 5.3.
 - 5.5 How many sequences are there that are both arithmetic and geometric? Explain.
6. Can you think of an example of a sequence which is arithmetic, geometric and Fibonacci? Explain.

7. Did the use of visualisation help you to do the worksheets on geometric sequences? Explain.
8. Did exploring patterns help you to do the worksheets on geometric sequences? Explain.
9. Do you think the topic of geometric sequences is relevant? Explain.
10. Give a summary of everything we have done so far.

Name: _____

1. What is the difference between a sequence and a series?

2. Did the visual approach help you to appreciate the summation formulae for
 - 2.1 arithmetic series? Explain.

 - 2.2 geometric series? Explain.

3. Are $T_k = 2 \cdot 3^{k-1}$, $T_k = 6^k$ and $T_k = 5 \cdot 7^{-k}$ all formulae for general terms of a geometric sequences? Explain.

4.
 - 4.1 Consider the series:

$$\sum_{k=1}^{10} (2k-1); \quad \sum_{r=2}^{\infty} 3 \cdot 2^r; \quad \sum_{s=1}^n s^2$$
 Explain what the following symbols represent here:
 - 4.1.1 \sum

 - 4.1.2 $k = 1$

 - 4.1.3 $r = 2$

 - 4.1.4 $s = 1$

$$4.1.5 \quad \infty$$

$$4.1.6 \quad n$$

$$4.1.7 \quad 2k - 1$$

$$4.1.8 \quad 3 \cdot 2^r$$

$$4.1.9 \quad s^2$$

4.2 Which of the above series is arithmetic? Why?

4.3 Which of the above series is geometric? Why?

5. Do you think it is ever possible to evaluate

$$5.1 \quad \sum_{k=1}^{\infty} [a + (k-1)d]? \quad \text{Explain.}$$

$$5.2 \quad \sum_{k=1}^{\infty} ar^{k-1}? \quad \text{Explain.}$$

6.

6.1 Are there an infinite number of real numbers?

6.2 Are there an infinite number of real numbers between 2 and 3?

6.3 Compare your answers in 6.1 and 6.2.

6.4 Do you think that there are the same number of real numbers as there are real numbers between 2 and 3?

Questions on Sequences and Series

Questionnaire 5

Name: _____

Please answer the following questions in the spaces provided:

1.
 - 1.1 What is the limit of a sequence?
 - 1.2 Give an example of a sequence whose terms approach infinity.
 - 1.3 Give an example of a sequence whose terms approach a fixed numerical value.
2.
 - 2.1 What did we find when we calculated $\lim_{k \rightarrow \infty} \frac{T_{k+1}}{T_k}$ for the Fibonacci sequence 1; 1; 2; 3; 5; 8; on the computer?
 - 2.2 Did the computer values for this and other sequences help you to appreciate the idea of a limit?
3.
 - 3.1 What is the limit of a series?

3.2.1 Did the below computer values help you to visualise that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1? \quad \text{Explain.}$$

0.5
0.75
0.875
0.9375
0.96875
0.984375
0.9921875
0.99609375
0.998046875
0.999023438

3.2.2 Did the blocks represented below help you to visualise that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1? \quad \text{Explain.}$$

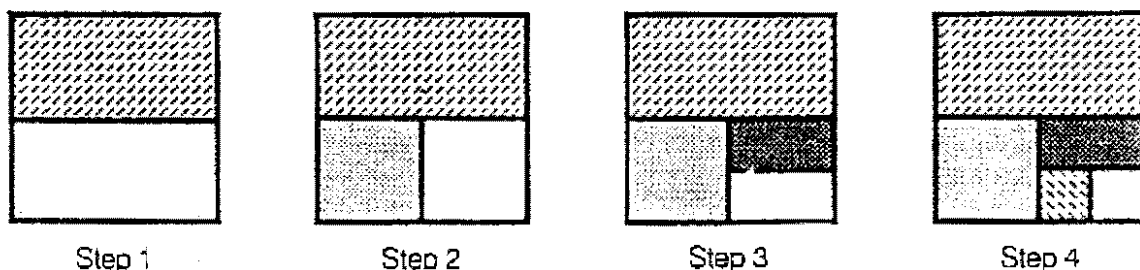


Fig. 13

(Bennett 1989: 134)

3.2.3 Did the graphical method help you to visualise that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1? \quad S_n \text{ is plotted against } n.$$

3.3 Which of the methods in 3.2 i.e. 3.2.1; 3.2.2; 3.2.3 did you find the most effective in illustrating that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$? Explain.

3.4 Do you think it is helpful to be provided with various means of visual illustrations such as in example 3.2 above?

4. Does the sum of terms of an arithmetic series approach a limit? Explain.
5. Do the terms of a geometric series approach a limit? Explain.
6. Are there an infinite number of real numbers as well as an infinite number of real numbers between -1 and 1 ? Explain.

Questions on Sequences and Series

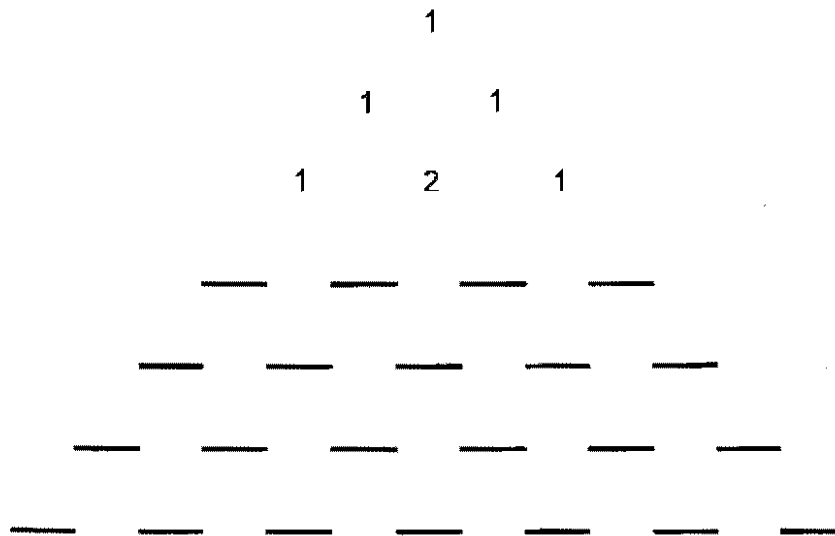
Questionnaire 6

Name: _____

Please answer the following questions in the spaces provided:

1.

1.1 Complete the next four rows of Pascal's triangle below:



1.2 Describe how Pascal's triangle is formed.

1.3 Do you think the knowledge of Pascal's triangle is useful in mathematics? Explain.

2.

2.1 Try to complete Pascal's triangle in the following way?



2.2 Which of the ways 1.1 & 1.2 above depicting Pascal's triangle is more effective? (Q1.1 or Q1.2) Explain.

3. Do you think it is necessary to prove a statement regarding positive

integers if you have shown it is true for about 10 values? Explain.

4.

4.1 How do you perform proofs by mathematical induction and when is it used?

4.2 Do you think mathematical induction is an acceptable method of proof? Explain.

4.3

4.3.1 Did the domino illustration help you to understand proof by induction? Explain.

4.3.2 Did the block illustration help you to understand proof by mathematical induction? Explain.

4.3.3 Which of 4.3.1 and 4.3.2 gave you a better idea of the meaning of proof by mathematical induction?

5. Has the study of patterns helped you

5.1 understand the concepts we have learnt?

5.2 see the need to establish rules?

6. Did the provision of visual illustrations help you to

6.1 understand the concepts we have learnt?

6.2 see the need to establish rules?

Questions on Sequences and Series

Questionnaire 7

Name: _____

Please answer the following questions in the spaces provided:

1. Is it necessary to establish rules in mathematics? Explain.

2. Is it necessary to prove results in mathematics? Explain.

3. Did the study of patterns help you

3.1. understand the concepts we have learnt? Explain.

3.2. see the need to establish rules? Explain.

3.3. see the need to prove results in mathematics? Explain.

4. Did the provision of visual illustrations help you to

4.1. understand the concepts we have learnt? Explain.

4.2. see the need to establish rules? Explain.

4.3. see the need to prove results in mathematics? Explain.

5. What sort of visual illustrations

5.1. helped you to understand the meaning of a concept? Explain.

5.2. did not help you to understand the meaning of a concept? Explain.

6. Do you think

6.1. discussing problems with your fellow students helps you to understand concepts? Explain.

6.2. establishing rules for yourselves gives you a better understanding of a topic in mathematics? Explain.

7. What did you find to be

7.1. the easiest thing we studied? Explain.

7.2. the most difficult thing we studied? Explain.

8. Did the study of patterns give you insight into the topic of sequences and series? Explain.

9. Did the use of visual illustrations give you insight into the topic of sequences and series?